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Volume 2, Number 2

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SPRINGER-VERLAG
BERLIN-GÖTTINGEN-HEIDELBERG
(Postverlagsort Berlin · 31.10.1958)

Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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Über die Potentialfunktionen von Kurvenbelegungen

CLAUS MÜLLER & ROLF LEIS

Das Verhalten der Potentiale einfacher Kurvenbelegungen bei Annäherung des Aufpunktes an die Kurve wurde zuerst von H. POINCARÉ [*1*] behandelt und am Beispiel eines homogenen gradlinigen bzw. kreisförmigen Drahtes explizit angegeben. Da diese Frage in der Zwischenzeit nicht weiter behandelt wurde, erscheint es reizvoll, die Problemstellung wieder aufzugreifen und auf das Verhalten der Potentiale von Dipolbelegungen auszudehnen.

Ziel der vorliegenden Arbeit ist es daher, das singuläre Verhalten der Potentiale von beliebigen Kurvenbelegungen bei Annäherung des Aufpunktes an die Kurve zu beschreiben. Dazu stellen wir die Laplacesche Potentialgleichung $\Delta U = 0$, der die Potentialfunktionen genügen, an den Ausgangspunkt der Überlegungen und bestimmen aus ihr durch Reihenansatz Lösungen, die ähnliches Verhalten zeigen wie die Potentialfunktionen. Durch einen Grenzübergang gelingt es dann mit Hilfe der Greenschen Formel zu zeigen, daß die so gewonnenen Lösungen bis auf eine überall stetige Funktion mit den Potentialen übereinstimmen. Man kann daher an den Potenzreihen das singuläre Verhalten der Potentialfunktionen ablesen. Die wichtigsten Ergebnisse lassen sich dann folgendermaßen formulieren:

Mit $\mathfrak{y}$ werde ein innerer Punkt der Kurve $\mathfrak{C}$ beschrieben. Es seien $\mathfrak{h}$ und $\mathfrak{b}$ die Vektoren der Haupt- und Binormalen. Dann verhält sich bei Annäherung an einen inneren Punkt $\mathfrak{y}$ der Kurve $\mathfrak{C}$ das Potential

$$\int\limits_{\mathfrak{C}} \varrho(s) \frac{ds}{|\mathfrak{x} - \mathfrak{y}(s)|}$$

der einfachen Belegung $\varrho(s)$ wie

$$-2\varrho(\mathfrak{y}) \lg|\mathfrak{y} - \mathfrak{x}| + O(1) \quad \text{für} \quad \mathfrak{x} \to \mathfrak{y}.$$

Gibt der Einheitsvektor

$$\mathfrak{d}(s) = \sin\alpha\, \mathfrak{h}(s) + \cos\alpha\, \mathfrak{b}(s)$$

die Richtung der zur Kurventangente senkrechten Achse des Dipols an, so stellt

$$\int\limits_{\mathfrak{C}} \varrho(s)\, \mathfrak{d}(s)\, \nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x} - \mathfrak{y}(s)|}\, ds$$

das Potential der Dipolbelegung dar. Diese Funktion verhält sich bei Annäherung an den inneren Punkt $\mathfrak{y}$ wie

$$2\varrho(\mathfrak{y}) \left(\mathfrak{d}(\mathfrak{y}) \frac{\mathfrak{x} - \mathfrak{y}}{|\mathfrak{x} - \mathfrak{y}|^2} \right) - \varrho(\mathfrak{y})\, \varkappa(\mathfrak{y})\, (\mathfrak{h}\, \mathfrak{d}) \lg|\mathfrak{x} - \mathfrak{y}| + O(1),$$

wobei $\varkappa(\mathfrak{y})$ die Krümmung im Punkte $\mathfrak{y}$ darstellt.

In § 1 der Arbeit wird zunächst das singuläre Verhalten der Potentialfunktionen von geraden Linien explizit angegeben. Da das kartesische Koordinatensystem zur Darstellung von beliebigen Kurven wenig geeignet ist, werden in § 2 krummlinige Koordinaten eingeführt, mit deren Hilfe dann in § 3 und § 4 das singuläre Verhalten der Potentialfunktionen von einfachen und doppelten Belegungen in inneren Punkten der Kurve behandelt wird. In § 5 wird abschließend noch das Verhalten an den Endpunkten nicht geschlossener Kurven untersucht.

Das in der vorliegenden Arbeit behandelte Problem zerfällt in zwei verschiedene Fragenkreise. Zunächst handelt es sich nämlich darum, den expliziten Ausdruck zu bestimmen, der das Verhalten in der Nähe der Kurve bestimmt. Dann ist zu prüfen, unter welchen Differenzierbarkeitsvoraussetzungen die gewonnenen Gesetzmäßigkeiten gelten.

Die untersuchten Kurven und ihre Belegungen werden dementsprechend zunächst als analytisch vorausgesetzt, um die formale Seite des Fragenkreises nicht mit Differenzierbarkeitsbetrachtungen zu belasten. Nach Erhalt dieser Ergebnisse ist es dann leicht, die Voraussetzungen auf Differenzierbarkeitsbedingungen endlich hoher Ordnung zu reduzieren.

§ 1. Vorbemerkungen

Unseren Betrachtungen liege der dreidimensionale euklidische Raum zugrunde. In ihm sei in einem kartesischen Koordinatensystem eine endliche doppelpunktfreie Kurve $\mathfrak{C}$ in der Form

$$\mathfrak{y} = \mathfrak{y}(s) \tag{1.1}$$

mit beliebig oft differenzierbarem $\mathfrak{y}(s)$ gegeben. Der Parameter s sei die Bogenlänge dieser Kurve. Der Punkt $\mathfrak{x}$ liege nicht auf $\mathfrak{C}$. Dann verstehen wir unter der Potentialfunktion $U_0(\mathfrak{x})$ einer einfachen Belegung $\varrho_0(s)$ der Kurve $\mathfrak{C}$ die Funktion

$$U_0(\mathfrak{x}) = \int\limits_{\mathfrak{C}} \frac{\varrho_0(s)\,ds}{|\mathfrak{x}-\mathfrak{y}|}\,. \tag{1.2}$$

Die Belegung $\varrho_0(s)$ wollen wir ebenfalls als beliebig oft differenzierbar voraussetzen. Es sei $\mathfrak{h}(s)$ der Hauptnormalen- und $\mathfrak{b}(s)$ der Binormalenvektor der Kurve $\mathfrak{C}$. Es sei ferner $\varrho_j(s)$ eine beliebig oft differenzierbare Funktion, dann erhalten wir durch

$$U_1(\mathfrak{x}) = \int\limits_{\mathfrak{C}} \varrho_1(s)\,\mathfrak{h}(s)\,\nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x}-\mathfrak{y}|}\,ds \tag{1.3}$$

bzw.

$$U_2(\mathfrak{x}) = \int\limits_{\mathfrak{C}} \varrho_2(s)\,\mathfrak{b}(s)\,\nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x}-\mathfrak{y}|}\,ds \tag{1.4}$$

die Potentiale der Dipolbelegungen. Dabei ist

$$\nabla = \frac{\partial}{\partial x}\,\mathfrak{e}_x + \frac{\partial}{\partial y}\,\mathfrak{e}_y + \frac{\partial}{\partial z}\,\mathfrak{e}_z \tag{1.5}$$

der „Nabla"-Operator. Durch das Symbol $\nabla_{\mathfrak{y}}$ deuten wir an, daß bezüglich $\mathfrak{y}$ differenziert wird. Es gilt

$$\nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x}-\mathfrak{y}|} = -\nabla_{\mathfrak{x}} \frac{1}{|\mathfrak{x}-\mathfrak{y}|}. \tag{1.6}$$

Die Funktionen $U_j(\mathfrak{x})$ $(j=0, 1, 2)$ sind für alle $\mathfrak{x}$, die nicht auf der Kurve liegen, differenzierbar. Bezeichnen wir mit

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.7}$$

den „Laplace"-Operator, so folgt für alle Punkte $\mathfrak{x} \notin \mathfrak{C}$

$$\Delta U_j(\mathfrak{x}) = 0. \tag{1.8}$$

Unsere Potentialfunktionen genügen also der Laplaceschen Differentialgleichung $\Delta U = 0$.

Um einen ersten Einblick in das Verhalten der Potentialfunktionen bei Annäherung von $\mathfrak{x}$ an die Kurve $\mathfrak{C}$ zu erhalten, betrachten wir das Potential $U_0(\mathfrak{x})$ für die Gerade $y=0$, $z=0$, $0 \leqq x \leqq L$ mit konstanter Belegung ϱ_0. Wir führen in der (y, z)-Ebene durch

$$y = r \sin\varphi; \qquad z = r\cos\varphi. \tag{1.9}$$

Polarkoordinaten ein und erhalten

$$\begin{aligned} U_0(\mathfrak{x}) &= \varrho_0 \int_0^L \frac{d\xi}{\sqrt{(\xi-x)^2+r^2}} \\ &= \varrho_0 \ln\left[\left(L-x+\sqrt{(L-x)^2+r^2}\right)\left(x+\sqrt{x^2+r^2}\right)\right] - 2\varrho_0 \ln r. \end{aligned} \tag{1.10}$$

Für $0 < x < L$ folgt aus (1.10), daß $U_0(\mathfrak{x})$ bei Annäherung von $\mathfrak{x}$ an die Kurve $\mathfrak{C}$ wie

$$-2\varrho_0 \ln r \tag{1.11}$$

singulär wird. Für $U_1(\mathfrak{x})$ und $U_2(\mathfrak{x})$ erhalten wir

$$U_1(\mathfrak{x}) = \varrho_1 \int_0^L \frac{r\sin\varphi}{\sqrt{(\xi-x)^2+r^2}^{\,3}}\, d\xi = \varrho_1 \frac{\sin\varphi}{r}\left\{\frac{L-x}{\sqrt{(L-x)^2+r^2}} + \frac{x}{\sqrt{x^2+r^2}}\right\} \tag{1.12}$$

und

$$U_2(\mathfrak{x}) = \varrho_2 \int_0^L \frac{r\cos\varphi}{\sqrt{(\xi-x)^2+r^2}^{\,3}}\, d\xi = \varrho_2 \frac{\cos\varphi}{r}\left\{\frac{L-x}{\sqrt{(L-x)^2+r^2}} + \frac{x}{\sqrt{x^2+r^2}}\right\}. \tag{1.13}$$

Daraus folgt, daß die Funktionen $U_1(\mathfrak{x})$ bzw. $U_2(\mathfrak{x})$ für $0 < x < L$ wie

$$2\varrho_1 \frac{\sin\varphi}{r} \quad \text{bzw.} \quad 2\varrho_2 \frac{\cos\varphi}{r} \tag{1.14}$$

singulär werden.

Wir werden erwarten, daß diese Ergebnisse auch bei beliebig geformten Kurven und Belegungen die Anfangsglieder einer Reihenentwicklung der Potentialfunktionen darstellen. Um das zu zeigen und um weitere Glieder dieser Entwicklung anzugeben, wollen wir die Kurve zunächst in einem geeigneten Koordinatensystem darstellen und formale Beziehungen in diesem Koordinatensystem ableiten.

§ 2. Das Koordinatensystem

Wir führen in der Normalenebene der Kurve $\mathfrak{C}$ Polarkoordinaten r, φ ein und können dann durch

$$\mathfrak{x}(s,\varphi,r) = \mathfrak{y}(s) + r\sin\varphi\,\mathfrak{h}(s) + r\cos\varphi\,\mathfrak{b}(s) \tag{2.1}$$

einen Punkt $\mathfrak{x}$ im Raum festlegen. Um ein Orthogonalsystem zu erhalten, setzen wir noch [$\tau(s)$ sei die Windung von $\mathfrak{C}$]

$$T(s) = \int_{s_0}^{s} \tau(t)\,dt \tag{2.2}$$

und verwenden fortan die Koordinaten

$$u^1 = s; \qquad u^2 = \varphi - T(s); \qquad u^3 = r. \tag{2.3}$$

Aus den Frenetschen Gleichungen

$$\begin{pmatrix} \mathfrak{t}' \\ \mathfrak{h}' \\ \mathfrak{b}' \end{pmatrix} = \begin{pmatrix} & \varkappa & \\ -\varkappa & & \tau \\ & -\tau & \end{pmatrix} \begin{pmatrix} \mathfrak{t} \\ \mathfrak{h} \\ \mathfrak{b} \end{pmatrix} \tag{2.4}$$

folgt wegen

$$\frac{\partial}{\partial u^1} = \frac{\partial}{\partial s} + \tau\frac{\partial}{\partial\varphi}; \qquad \frac{\partial}{\partial u^2} = \frac{\partial}{\partial\varphi}; \qquad \frac{\partial}{\partial u^3} = \frac{\partial}{\partial r} \tag{2.5}$$

unmittelbar mit $\delta = 1 - r\varkappa\sin\varphi$

$$\begin{aligned} \mathfrak{x}_{|1} &= \frac{\partial}{\partial u^1}\mathfrak{x} = \delta\mathfrak{t}, \\ \mathfrak{x}_{|2} &= \frac{\partial}{\partial u^2}\mathfrak{x} = r\cos\varphi\,\mathfrak{h} - r\sin\varphi\,\mathfrak{b}, \\ \mathfrak{x}_{|3} &= \frac{\partial}{\partial u^3}\mathfrak{x} = \sin\varphi\,\mathfrak{h} + \cos\varphi\,\mathfrak{b}, \end{aligned} \tag{2.6}$$

so daß sich

$$g_{ik} = \begin{pmatrix} \delta^2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

ergibt. In einer Umgebung der Kurve $\mathfrak{C}$ ist die Darstellung (2.1) umkehrbar eindeutig, wenn dort $g = \det(g_{ik})$ nicht verschwindet. Fordern wir daher

$$r > 0; \qquad r\varkappa\sin\varphi < 1, \tag{2.8}$$

so ist diese Bedingung erfüllt. Wir wollen unsere weiteren Überlegungen deshalb auf das schlauchartige Gebiet $\mathfrak{G}$ beschränken, welches die Kurve $\mathfrak{C}$ und alle Punkte, die der Bedingung (2.8) genügen, enthält.

Wir betrachten nun die in $\mathfrak{G}$ enthaltene Koordinatenfläche $u^3 = r = \text{const.}$ Der Maßtensor dieser Fläche Z hat die Gestalt

$$\gamma_{\alpha\beta} = \begin{pmatrix} \delta^2 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{2.9}$$

und das Flächenelement ergibt sich wegen

$$\frac{\partial(u^1, u^2)}{\partial(s, \varphi)} = \begin{vmatrix} 1 & -\tau \\ 0 & 1 \end{vmatrix} = 1 \tag{2.10}$$

zu

$$dF = r\,\delta\, du^1\, du^2 = r\,\delta\, ds\, d\varphi. \tag{2.11}$$

Den Normalenvektor $\mathfrak{n}$ der Fläche Z erhalten wir durch

$$\mathfrak{n} = \frac{1}{\sqrt{\det \gamma_{\alpha\beta}}}\, \mathfrak{x}_{|1} \times \mathfrak{x}_{|2} = \sin\varphi\, \mathfrak{h} + \cos\varphi\, \mathfrak{b} = \mathfrak{x}_{|3}. \tag{2.12}$$

Es sei $g(\mathfrak{x})$ eine differenzierbare Funktion der Ortskoordinaten. Dann gilt auf Z

$$\frac{\partial}{\partial n}\, g(\mathfrak{x}) = \mathfrak{n}\, \nabla g = \frac{\partial g}{\partial u^3} = \frac{\partial g}{\partial r}. \tag{2.13}$$

Schließlich geben wir noch den Δ-Operator im u-System an. Es ist

$$\begin{aligned} \Delta U &= \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial u^i}\, g^{ik} \sqrt{g}\, \frac{\partial U}{\partial u^k} \\ &= \frac{1}{\delta}\, \frac{\partial}{\partial u^1}\, \frac{1}{\delta}\, \frac{\partial}{\partial u^1}\, U + \frac{1}{r^2 \delta}\, \frac{\partial}{\partial u^2}\, \delta\, \frac{\partial}{\partial u^2}\, U + \frac{1}{r\,\delta}\, \frac{\partial}{\partial u^3}\, r\,\delta\, \frac{\partial}{\partial u^3}\, U \\ &= U_{|1}\, \frac{r}{\delta^3}\, (\varkappa' \sin\varphi + \varkappa\,\tau \cos\varphi) - \frac{\varkappa \cos\varphi}{r\,\delta}\, U_{|2} - \frac{\varkappa \sin\varphi}{\delta}\, U_{|3} + \\ &\quad + \frac{U_{|11}}{\delta^2} + \frac{U_{|22}}{r^2} + \frac{1}{r}\, \frac{\partial}{\partial r}\, r\, U_{|3}. \end{aligned} \tag{2.14}$$

Wir wollen aus (2.14) durch Reihenansatz in φ periodische Lösungen bestimmen, deren erste Glieder ähnliche Gestalt haben, wie wir es uns am Falle der geraden Linie im § 1 überlegt haben und erhalten

$$\begin{aligned} V_0(\mathfrak{x}) &= -2\varrho_0(s) \lg r - \varkappa\, \varrho_0 \sin\varphi\, r \lg r + \frac{1}{2}\, r^2 \lg r\, (\varrho_0'' - \varkappa^2 \varrho_0 \sin^2\varphi) \\ V_1(\mathfrak{x}) &= \frac{2}{r}\, \varrho_1(s) \sin\varphi + \varrho_1\, \varkappa \sin^2\varphi - \\ &\quad - r \lg r \left\{\sin\varphi \left(\varrho_1'' + \frac{\varrho_1 \varkappa^2}{4} - \varrho_1 \tau^2\right) + \cos\varphi\, (\varrho_1 \tau' + 2\tau\, \varrho_1')\right\} + \\ &\quad + \frac{3}{4}\, r\, \varkappa^2 \varrho_1 \sin^3\varphi + \frac{1}{4}\, r^2 \lg r\, \varkappa \left(-\varrho_1'' - \frac{\varrho_1 \varkappa^2}{4} + \varrho_1 \tau^2\right) \\ V_2(\mathfrak{x}) &= \frac{2}{r}\, \varrho_2(s) \cos\varphi + \frac{1}{2}\, \varrho_2\, \varkappa \sin 2\varphi - \\ &\quad - r \lg r \left\{\cos\varphi \left(\varrho_2'' + \frac{\varrho_2 \varkappa^2}{4} - \varrho_2 \tau^2\right) - \sin\varphi\, (\varrho_2 \tau' + 2\tau\, \varrho_2')\right\} - \\ &\quad - \frac{3}{4}\, r\, \varkappa^2 \varrho_2 \cos^3\varphi + \frac{1}{4}\, r^2 \lg r\, \varkappa\, (\varrho_2 \tau' + 2\tau\, \varrho_2') \end{aligned} \tag{2.15}$$

mit

$$\Delta V_j(\mathfrak{x}) = O(1) \qquad \text{für kleine } r. \tag{2.16}$$

Im nächsten Paragraphen werden wir zeigen, daß unsere Potentialfunktionen $U_j(\mathfrak{x})$ sich für $r \to 0$ wie $V_j(\mathfrak{x})$ verhalten.

§ 3. Das singuläre Verhalten der Potentialfunktionen von einfachen Belegungen

Im letzten Paragraphen haben wir die Funktion $V_0(\mathfrak{x})$ eingeführt, die der Abschätzung

$$\Delta V_0(\mathfrak{x}) = O(1) \quad \text{für kleine } r \tag{3.1}$$

genügt. Um zu zeigen, daß diese Funktion sich in der Nähe der Kurve (für kleine r) wie $U_0(\mathfrak{x})$ verhält, führen wir folgende Bezeichnungen ein:

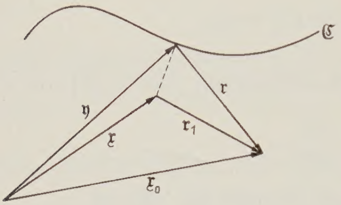

Fig. 1

Die Oberfläche unseres Gebietes $\mathfrak{G}$ sei Ω. $\mathfrak{G}$ enthalte das Gebiet $\mathfrak{Z}$ mit der Oberfläche Z ($u^3 = r =$ const). Dann wird das Gebiet $\mathfrak{B} = \mathfrak{G} - \mathfrak{Z}$ von den Oberflächen Ω und Z berandet. In $\mathfrak{B}$ wählen wir einen festen Punkt $\mathfrak{x}_0$ aus, um den wir eine genügend kleine Kugel $\mathfrak{K}$ mit dem Radius ϱ und der Oberfläche K legen. $\mathfrak{K}$ liege ganz in $\mathfrak{B}$. $\mathfrak{x}$ sei ein beliebiger Punkt in $\mathfrak{G}$, $R(\mathfrak{x}, \mathfrak{x}_0)$ der Abstand zwischen $\mathfrak{x}$ und $\mathfrak{x}_0$.

Wenden wir nun die Greensche Formel auf das Gebiet $\mathfrak{B} - \mathfrak{K}$ mit den Funktionen $V_0(\mathfrak{x})$ und $R^{-1}(\mathfrak{x}, \mathfrak{x}_0)$ an, dann erhalten wir

$$\begin{aligned}\int\limits_{\mathfrak{B}-\mathfrak{K}} \left(V_0 \Delta \frac{1}{R} - \frac{1}{R} \Delta V_0\right) dV = \int\limits_{\Omega} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial}{\partial n} V_0\right) dF - \\ - \int\limits_{Z} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial}{\partial n} V_0\right) dF - \int\limits_{K} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial}{\partial n} V_0\right) dF.\end{aligned} \tag{3.2}$$

Aus

$$\int\limits_{K} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial V_0}{\partial n}\right) dF = \int\limits_{K} \left(-\frac{V_0}{\varrho^2} - \frac{1}{\varrho} \frac{\partial V_0}{\partial \varrho}\right) \varrho^2 \, d\omega \tag{3.3}$$

folgt

$$\lim_{\varrho \to 0} \int\limits_{K} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial}{\partial n} V_0\right) dF = -4\pi V_0(\mathfrak{x}_0). \tag{3.4}$$

Da das Integral

$$\int\limits_{\mathfrak{K}} \left(V_0 \Delta \frac{1}{R} - \frac{1}{R} \Delta V_0\right) dV \tag{3.5}$$

existiert, erhalten wir aus (3.2)

$$\begin{aligned}-\int\limits_{\mathfrak{B}} \frac{1}{R} \Delta V_0 \, dV = 4\pi V_0(\mathfrak{x}_0) + \int\limits_{\Omega} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial V_0}{\partial n}\right) dF - \\ - \int\limits_{Z} \left(V_0 \frac{\partial}{\partial n} \frac{1}{R} - \frac{1}{R} \frac{\partial V_0}{\partial n}\right) dF.\end{aligned} \tag{3.6}$$

Mit $\mathfrak{r} = \mathfrak{x}_0 - \mathfrak{y}$ und (vgl. Fig. 1)

$$\mathfrak{r}_1 = \mathfrak{r} - r(\sin\varphi\, \mathfrak{h} + \cos\varphi\, \mathfrak{b}) = \mathfrak{r} - r\,\mathfrak{n} \tag{3.7}$$

folgt

$$R^2 = |\mathfrak{x}_0 - \mathfrak{y}|^2 - 2r\,\mathfrak{r}\,\mathfrak{n} + r^2, \tag{3.8}$$

und es gilt bei festem $\mathfrak{x}_0$ für kleine r

$$(3.9)\qquad \frac{1}{R} = \frac{1}{|\mathfrak{x}_0-\mathfrak{y}|} + \frac{r\,\mathfrak{x}\,\mathfrak{n}}{|\mathfrak{x}_0-\mathfrak{y}|^3} + O(r^2).$$

Wegen

$$(3.10)\qquad \frac{\mathfrak{x}\,\mathfrak{n}}{|\mathfrak{x}_0-\mathfrak{y}|^3} = \mathfrak{n}\,\nabla_{\mathfrak{y}}\frac{1}{|\mathfrak{x}_0-\mathfrak{y}|},$$

erhalten wir schließlich

$$(3.11)\qquad \begin{aligned} \frac{1}{R} &= \frac{1}{|\mathfrak{x}_0-\mathfrak{y}|} + r\,\mathfrak{n}\,\nabla\frac{1}{|\mathfrak{x}_0-\mathfrak{y}|} + O(r^2),\\ \frac{\partial}{\partial n}\frac{1}{R} &= \mathfrak{n}\,\nabla\frac{1}{|\mathfrak{x}_0-\mathfrak{y}|} + O(r). \end{aligned}$$

Aus der Darstellung (2.15) der Funktion $V_0(\mathfrak{x})$ entnehmen wir

$$(3.12)\qquad \begin{aligned} V_0 &= -\,2\varrho_0\lg r + o(1)\\ \frac{\partial V_0}{\partial n} &= -\,2\varrho_0\frac{1}{r} + O(\lg r). \end{aligned}$$

Zusammen mit (3.11) folgt daraus für $r\to 0$ nach (1.2)

$$(3.13)\qquad \int\limits_Z\left(V_0\frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial}{\partial n}V_0\right)dF = \int\limits_{\mathfrak{C}}\int\limits_0^{2\pi}\frac{2\varrho_0}{|\mathfrak{x}_0-\mathfrak{y}|}\,d\varphi\,ds + o(1) = 4\pi\,U_0(\mathfrak{x}_0) + o(1),$$

da wegen der vorausgesetzten endlichen Länge der Kurve alle anderen Größen bei festem $\mathfrak{x}_0$ beschränkt bleiben und bei nicht geschlossenen Kurven die Beiträge der Endflächen wegen $dF = r\,dr\,d\varphi$ von der Ordnung $o(1)$ sind.

Wegen

$$(3.14)\qquad \Delta V_0 = O(1)$$

existiert das Integral

$$(3.15)\qquad \int\limits_{\mathfrak{Z}}\frac{\Delta V_0}{R}\,dV$$

für alle $\mathfrak{x}_0$, und wir erhalten aus (3.6) durch den Grenzübergang $r\to 0$ für alle $\mathfrak{x}_0$, die nicht auf $\mathfrak{C}$ liegen,

$$(3.16)\qquad 4\pi\,U_0(\mathfrak{x}_0) = 4\pi\,V_0(\mathfrak{x}_0) + \int\limits_{\mathfrak{G}}\frac{\Delta V_0}{R}\,dV + \int\limits_{\Omega}\left(V_0\frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial}{\partial n}V_0\right)dF.$$

In (3.16) ist das Integral über Ω eine stetige Funktion für alle inneren Punkte $\mathfrak{x}$ von $\mathfrak{G}$. Da ΔV_0 überall in $\mathfrak{G}$ beschränkt ist, stellt auch das Integral über $\mathfrak{G}$ für alle $\mathfrak{x}\in\mathfrak{G}$ eine stetige Funktion dar [*2*]. Für alle inneren Punkte von $\mathfrak{G}$, die nicht auf $\mathfrak{C}$ liegen, stimmt das Potential $U_0(\mathfrak{x})$ daher bis auf eine überall stetige Funktion $f_0(\mathfrak{x})$ mit $V_0(\mathfrak{x})$ überein. Durch den Grenzübergang

$$(3.17)\qquad \lim_{\mathfrak{x}_\nu\to\mathfrak{x}_{\mathfrak{C}}}\{U_0(\mathfrak{x}_\nu) - V_0(\mathfrak{x}_\nu)\} = \lim_{\mathfrak{x}_\nu\to\mathfrak{x}_{\mathfrak{C}}} f_0(\mathfrak{x}_\nu) = f_0(\mathfrak{x}_{\mathfrak{C}})$$

erhalten wir daher

Satz 1. *Es sei*

$$V_0(\mathfrak{x}) = -2\varrho_0(s)\lg r - \varkappa\,\varrho_0 \sin\varphi\, r \lg r + \tfrac{1}{2} r^2 \lg r\,(\varrho_0'' - \varkappa^2 \varrho_0 \sin^2\varphi).$$

Dann gibt es eine überall in $\mathfrak{G}$ *stetige Funktion* $f_0(\mathfrak{x})$, *so daß für alle* $\mathfrak{x}$ *in* $\mathfrak{G}$

$$U_0(\mathfrak{x}) = V_0(\mathfrak{x}) + f_0(\mathfrak{x})$$

gilt.

Für $\mathfrak{x} \to \mathfrak{y} \in \mathfrak{C}$ verhält sich $U_0(\mathfrak{x})$ daher wie

$$-2\varrho_0(\mathfrak{y}) \lg|\mathfrak{x} - \mathfrak{y}|. \tag{3.18}$$

§ 4. Das singuläre Verhalten der Potentialfunktionen von Dipolbelegungen

Um das singuläre Verhalten der Potentiale von Dipolbelegungen anzugeben, gehen wir ähnlich vor, wie bei den einfachen Belegungen. Zur Vereinfachung der Rechnungen fassen wir die Funktionen $U_1(\mathfrak{x})$ und $U_2(\mathfrak{x})$ in

$$U = \sin\alpha\, U_1 + \cos\alpha\, U_2 = \int\limits_{\mathfrak{C}} \varrho(s)\,\mathfrak{d}(s)\,\nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x} - \mathfrak{y}(s)|}\, ds \tag{4.1}$$

mit $\mathfrak{d}(s) = \sin\alpha\,\mathfrak{h}(s) + \cos\alpha\,\mathfrak{b}(s)$ zusammen. Entsprechend betrachten wir

$$V = \sin\alpha\, V_1 + \cos\alpha\, V_2 \tag{4.2}$$

mit

$$\varDelta V = O(1) \tag{4.3}$$

und wollen zeigen, daß die Funktionen U und V bei Annäherung an die Kurve gleiches singuläres Verhalten aufweisen.

Dazu wenden wir analog (3.2ff.) die Greensche Formel mit der Funktion $V(\mathfrak{x})$ an und erhalten für alle $\mathfrak{x}_0 \notin \mathfrak{C}$

$$\begin{aligned} -\int\limits_{\mathfrak{B}} \frac{\varDelta V}{R}\, dV = 4\pi V(\mathfrak{x}_0) &+ \int\limits_{\Omega} \left(V_0 \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial V}{\partial n}\right) dF - \\ &- \int\limits_{Z} \left(V_0 \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial V}{\partial n}\right) dF. \end{aligned} \tag{4.4}$$

Nun gilt wegen (2.12), (3.11) und

$$\frac{\partial V}{\partial n} = -2\varrho\cos(\varphi - \alpha)\frac{1}{r^2} + O(\lg r) \tag{4.5}$$

für kleine r

$$\begin{aligned} \int\limits_{Z} \left(V \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial V}{\partial n}\right) dF &= 4 \iint\limits_{\mathfrak{C}\ 0}^{\ \ 2\pi} \varrho\cos(\varphi - \alpha)\,\mathfrak{n}\,\nabla \frac{1}{|\mathfrak{x}_0 - \mathfrak{y}|}\, d\varphi\, ds + \\ &+ \frac{2}{r} \iint\limits_{\mathfrak{C}\ 0}^{\ \ 2\pi} \varrho\cos(\varphi - \alpha) \frac{1}{|\mathfrak{x}_0 - \mathfrak{y}|}\, d\varphi\, ds - \\ &- 2 \iint\limits_{\mathfrak{C}\ 0}^{\ \ 2\pi} \varrho\cos(\varphi - \alpha)\sin\varphi\,\varkappa \frac{1}{|\mathfrak{x}_0 - \mathfrak{y}|}\, d\varphi\, ds + o(1) \end{aligned} \tag{4.6}$$

oder

$$(4.7)\qquad \int\limits_Z \left(V \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial V}{\partial n}\right) dF = 4\pi\, U(\mathfrak{x}_0) - 2\pi \sin\alpha\, U_0(\mathfrak{x}_0, \varkappa\varrho_1) + o(1),$$

wobei wir unter $U_0(\mathfrak{x}, \varkappa\varrho_1)$ das Potential mit der einfachen Belegung $\varkappa\varrho_1$ verstehen. Durch den Grenzübergang $r \to 0$ folgt daher aus (4.4)

$$(4.8)\qquad \begin{aligned} 4\pi\, U(\mathfrak{x}_0) &= 4\pi\, V(\mathfrak{x}_0) + 2\pi \sin\alpha\, U_0(\mathfrak{x}_0, \varkappa\varrho_1) + \\ &\quad + \int\limits_{\mathfrak{G}} \frac{\Delta V}{R}\, dV + \int\limits_{\Omega} \left(V \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial}{\partial n} V\right) dF. \end{aligned}$$

Die Integrale stellen wieder stetige Funktionen dar, es gilt daher

Satz 2. *Es sei*

$$V(\mathfrak{x}) = \sin\alpha\, V_1(\mathfrak{x}) + \cos\alpha\, V_2(\mathfrak{x}) = \frac{2\varrho(s)}{r}(\sin\alpha + \cos\alpha) + \cdots$$

die durch Gl. (2.15) *definierte Funktion. Dann gibt es eine überall in* $\mathfrak{G}$ *stetige Funktion* $f(\mathfrak{x})$, *so daß für alle* $\mathfrak{x}$ *in* $\mathfrak{G}$

$$U(\mathfrak{x}) = V(\mathfrak{x}) + \tfrac{1}{2}\, U_0(\mathfrak{x}, \varkappa\varrho_1) \sin\alpha + f(\mathfrak{x})$$

gilt.

Bei Annäherung des Punktes $\mathfrak{x}$ an den Punkt $\mathfrak{y}$ der Kurve $\mathfrak{C}$ wird die Potentialfunktion

$$(4.9)\qquad U(\mathfrak{x}) = \int\limits_{\mathfrak{C}} \varrho(s)\, \mathfrak{d}(s)\, \nabla_{\mathfrak{y}} \frac{1}{|\mathfrak{x} - \mathfrak{y}|}\, ds,$$

daher wie

$$(4.10)\qquad \varrho(\mathfrak{y}) \left\{ \frac{2\vartheta(\mathfrak{y})\,(\mathfrak{x} - \mathfrak{y})}{|\mathfrak{x} - \mathfrak{y}|^2} - \varkappa(\mathfrak{y})\,(\mathfrak{h}\,\mathfrak{d}) \lg|\mathfrak{x} - \mathfrak{y}| \right\}$$

singulär.

§ 5. Das Verhalten der Potentialfunktionen an den Kurvenenden

Wir haben bisher das singuläre Verhalten der Potentialfunktionen bei Annäherung des Aufpunktes $\mathfrak{x}$ an einen inneren Punkt der Kurve $\mathfrak{C}$ diskutiert. Wir setzen nun voraus, daß $\mathfrak{C}$ nicht geschlossen ist. Die Endpunkte seien durch $s = 0$ bzw. $s = L$ gegeben. Wir wollen das Verhalten bei Annäherung von $\mathfrak{x}$ an einen dieser Punkte bestimmen und setzen zur Abkürzung $\mathfrak{x}(0) = \mathfrak{o}$.

Für alle Punkte $\mathfrak{x} \in \mathfrak{G}$ haben wir die Gleichung

$$(5.1)\qquad U_0(\mathfrak{x}) = V_0(\mathfrak{x}) + f_0(\mathfrak{x})$$

mit

$$(5.2)\qquad f_0(\mathfrak{x}) = \frac{1}{4\pi}\left\{ \int\limits_{\mathfrak{G}} \frac{\Delta V_0}{R}\, dV + \int\limits_{\Omega} \left(V_0 \frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial}{\partial n} V_0\right) dF \right\}$$

bewiesen. Es sei $\mathfrak{x}_0$ ein Punkt mit den Koordinaten $(0, r, \varphi)$. $\mathfrak{x}_0$ liegt also auf der Grundfläche K der schlauchartigen Fläche Ω. Wir können das Flächenintegral über Ω in Integrale über K und über $\Omega^* = \Omega - K$ zerlegen. Die Fläche

K $(s=0;\ 0\leqq r\leqq c;\ 0\leqq\varphi\leqq 2\pi)$ ist eben. Ihr Flächenelement lautet

$$dF = r\,dr\,d\varphi. \tag{5.3}$$

Auf K gilt

$$\frac{\partial}{\partial n} = -\frac{1}{\delta}\,\frac{\partial}{\partial s}. \tag{5.4}$$

Zur Diskussion des Verhaltens an den Kurvenenden führen wir zunächst in (5.1) den Grenzübergang $\mathfrak{x}\to\mathfrak{x}_0$ durch und gehen anschließend mit $\mathfrak{x}_0$ gegen $\mathfrak{o}$. Bei beiden Grenzübergängen bleiben das Gebietsintegral und das Flächenintegral über Ω^* beschränkt, da sie für $\mathfrak{x}\in K$ stetige Funktionen darstellen. Wir diskutieren daher die Integrale

$$\int\limits_K \left(\frac{1}{\delta}\,\frac{\partial}{\partial s}\,V_0\right)\frac{1}{R}\,dF \quad\text{und}\quad \int\limits_K V_0\,\frac{\partial}{\partial n}\,\frac{1}{R}\,dF. \tag{5.5}$$

Die Integrale in (5.5) sind Potentialfunktionen mit einfacher bzw. doppelter Flächenbelegung. Das Potential mit der einfachen Belegung $\frac{1}{\delta}\cdot\frac{\partial}{\partial s}\,V_0$ bleibt beim Grenzübergang $\mathfrak{x}\to\mathfrak{x}_0\neq\mathfrak{o}$ stetig, da die Belegung in einer Umgebung des Punktes $\mathfrak{x}_0$ beschränkt ist [3]. Wir erhalten also

$$\lim_{\mathfrak{x}\to\mathfrak{x}_0}\int\limits_K \frac{1}{\delta}\,\frac{\partial}{\partial s}\,V_0\,\frac{1}{R}\,dF = \int\limits_K \frac{1}{\delta}\,\frac{\partial}{\partial s}\,V_0\,\frac{1}{R_0}\,dF \tag{5.6}$$

und wollen zeigen, daß das Integral auf der rechten Seite von (5.6) für alle $\mathfrak{x}_0$ (auch für $\mathfrak{x}_0=\mathfrak{o}$) beschränkt bleibt. Es gilt

$$\left|\int\limits_K \frac{1}{\delta}\,\frac{\partial}{\partial s}\,V_0\,\frac{1}{R_0}\,dF\right| \leqq A_1 + A_2\left|\int\limits_K \frac{\lg r\,dF}{|\mathfrak{x}-\mathfrak{x}_0|}\right|. \tag{5.7}$$

Nun ist mit $|\mathfrak{x}_0| = r_0$

$$\left|\int\limits_K \frac{\lg r\,dF}{|\mathfrak{x}-\mathfrak{x}_0|}\right| \leqq \int\limits_{|r-r_0|\leqq\frac{1}{2}r_0} \frac{|\lg r|}{|r-r_0|}\,dF + \int\limits_{|r-r_0|\leqq\frac{1}{2}r_0} \frac{|\lg r|}{|r-r_0|}\,dF \tag{5.8}$$

und

$$\begin{aligned} &\int\limits_{|r-r_0|\leqq\frac{1}{2}r_0} \frac{|\lg r|}{|r-r_0|}\,dF \leqq A_3\,r_0\,|\lg r_0|\,,\\ &\int\limits_{|r-r_0|\leqq\frac{1}{2}r_0} \frac{|\lg r|}{|r-r_0|}\,dF \leqq 2\pi\int\limits_{\frac{1}{2}r_0}^{C} |\lg|r_0-\varrho||\,d\varrho \leqq A_4 + A_5\,r_0\,|\lg r_0|. \end{aligned} \tag{5.9}$$

Wir erhalten also

$$\left|\int\limits_K \frac{1}{\delta}\,\frac{\partial}{\partial s}\,V_0\,\frac{1}{R_0}\,dF\right| \leqq B_1 + B_2\,r_0\,|\lg r_0| \leqq C. \tag{5.10}$$

Für das Integral mit der Dipolbelegung V_0 folgt beim Grenzübergang $\mathfrak{x}\to\mathfrak{x}_0$ [3]

$$\lim_{\mathfrak{x}\to\mathfrak{x}_0}\int\limits_K V_0\,\frac{\partial}{\partial n}\,\frac{1}{R}\,dF = -2\pi\,V_0(\mathfrak{x}_0). \tag{5.11}$$

Es folgt also

$$U_0(\mathfrak{x}_0) = \tfrac{1}{2} V_0(\mathfrak{x}_0) + g(\mathfrak{x}_0) \tag{5.12}$$

mit

$$|g(\mathfrak{x}_0)| \leqq K < \infty \qquad \text{für alle } \mathfrak{x}_0. \tag{5.13}$$

Für $\mathfrak{x}_0 \to \mathfrak{o}$ verhält sich die Potentialfunktion $U_0(\mathfrak{x}_0)$ daher wie $\frac{1}{2} V_0(\mathfrak{x}_0)$.

Für die Potentialfunktionen mit Dipolbelegungen $U_1(\mathfrak{x})$ und $U_2(\mathfrak{x})$ gelten entsprechende Überlegungen. An Stelle von (5.7) erhalten wir jetzt

$$\left|\int\limits_K \left(\frac{1}{\delta}\,\frac{\partial}{\partial s} V\right)\frac{1}{R_0}\,dF\right| \leqq A_1 + A_2\left|\int\limits_K \frac{\sin\varphi\,dF}{r\,|\mathfrak{x}-\mathfrak{x}_0|}\right| + A_3\left|\int\limits_K \frac{\cos\varphi\,dF}{r\,|\mathfrak{x}-\mathfrak{x}_0|}\right|. \tag{5.14}$$

Die beiden Integrale auf der rechten Seiten von (5.14) verschwinden aus Symmetriegründen. Es gilt z.B.

$$\int\limits_K \frac{\sin\varphi\,dF}{r\,|\mathfrak{x}-\mathfrak{x}_0|} = \int\limits_K \frac{y\,dx\,dy}{(x^2+y^2)\sqrt{(x-x_0)^2+y^2}} = 0, \tag{5.15}$$

und es folgt, daß die Potentialfunktion $U(\mathfrak{x}_0)$ für $\mathfrak{x}_0 \to \mathfrak{o}$ wie

$$\tfrac{1}{2}\left\{V(\mathfrak{x}_0) + \tfrac{1}{2} U_0(\mathfrak{x}_0, \varkappa\,\varrho_1)\sin\alpha\right\} \tag{5.16}$$

singulär wird. Wir erhalten also

Satz 3. *Es sei $U(\mathfrak{x})$ das Potential einer einfachen oder doppelten Belegung der nicht geschlossenen Kurve $\mathfrak{C}$. Bei Annäherung an innere Punkte von $\mathfrak{C}$ verhalte sich $U(\mathfrak{x})$ wie $V(\mathfrak{x})$, wobei $V(\mathfrak{x})$ eine der in Satz 2 bzw. Satz 3 genannten Funktionen darstellt. Liegt der Punkt $\mathfrak{x}$ in der Normalenebene durch ein Kurvenende, so wird das Verhalten des Potentials $U(\mathfrak{x})$ bei Annäherung an den Endpunkt durch die Funktion $\frac{1}{2}V(\mathfrak{x})$ beschrieben.*

§ 6. Abschwächung der Differenzierbarkeitsvoraussetzungen

Zur einfacheren Formulierung der Ergebnisse haben wir bisher von der Kurve $\mathfrak{C}$ und den Belegungsfunktionen ϱ beliebig hohe Differenzierbarkeit vorausgesetzt. Da wir diese Voraussetzung jedoch nicht benutzt haben, wollen wir sie nun auf Differenzierbarkeitsbedingungen endlich hoher Ordnung abschwächen.

Bei der Bildung der Funktion $f_j(\mathfrak{x})$ sind die Größen ϱ_j^{IV} und τ^{III} aufgetreten. Höhere Ableitungen wurden niemals benötigt. Es folgt daher, daß unsere Ergebnisse gelten, wenn wir die Kurve $\mathfrak{C}$ als sechsmal und die Belegungsfunktionen ϱ_j als viermal stetig differenzierbar voraussetzen. Wenn wir auf die Stetigkeit der Funktionen $f_j(\mathfrak{x})$ verzichten und nur noch ihre gleichmäßige Beschränktheit zeigen, lassen sich diese Voraussetzungen weiter verringern. Wir beweisen

Satz 4. *Die Kurve $\mathfrak{C}$ sei dreimal stetig differenzierbar. Die Belegung $\varrho_0(s)$ sei hölderstetig. Dann bleibt in jedem ganz in $\mathfrak{G}$ enthaltenen abgeschlossenen Teilbereich $\mathfrak{G}^*$*

$$U_0(\mathfrak{x}) + 2\varrho_0(s)\lg|\mathfrak{x}-\mathfrak{y}|$$

gleichmäßig beschränkt,

und

Satz 5. *Die Kurve $\mathfrak{C}$ sei viermal stetig differenzierbar. Die Belegung $\varrho(s)$ sei hölderstetig differenzierbar. Dann bleibt in jedem ganz in $\mathfrak{G}$ enthaltenen abgeschlossenen Teilbereich $\mathfrak{G}^*$*

$$U(\mathfrak{x}) - 2\varrho(\mathfrak{y}) \frac{\mathfrak{d}(\mathfrak{y})\,(\mathfrak{x}-\mathfrak{y})}{|\mathfrak{x}-\mathfrak{y}|^2} + \varrho(\mathfrak{y})\,\varkappa(\mathfrak{y})\,(\mathfrak{h}\,\mathfrak{d}) \lg|\mathfrak{x}-\mathfrak{y}|$$

gleichmäßig beschränkt.

Wir beweisen zunächst Satz 3 für konstante Belegungen $\varrho_0(s)$. Dazu setzen wir

$$V_0^*(\mathfrak{x}) = -2\varrho_0 \lg r. \tag{6.1}$$

$V_0^*(\mathfrak{x})$ genügt für kleine r der Abschätzung

$$\varDelta V_0^*(\mathfrak{x}) = O\left(\frac{1}{r}\right). \tag{6.2}$$

Es existiert daher das Integral

$$\int\limits_{\mathfrak{G}} \frac{\varDelta V_0^*}{R}\, dV = O\left(\int\limits_{\mathfrak{G}} \frac{\delta\, dr\, ds\, d\varphi}{R}\right) = O\left(\int\limits_0^{2\pi}\left(\int\limits_F \frac{dF}{R}\right) d\varphi\right), \tag{6.3}$$

so daß unsere Überlegungen im § 3 richtig bleiben, wenn wir $V_0^*(\mathfrak{x})$ statt $V_0(\mathfrak{x})$ verwenden. In $f_0^*(\mathfrak{x})$ gehen nur Ableitungen von $\mathfrak{C}$ bis zur Ordnung von $\varkappa$ ein. Bei der Einführung des u-Systems hatten wir jedoch τ benutzt, so daß $\mathfrak{C}$ als dreimal stetig differenzierbar vorausgesetzt werden muß. Satz 3 ist damit für konstante Belegungen bewiesen.

Es sei $\varrho_0(s)$ nun eine hölderstetige Funktion. Dann gilt in einem beliebigen Punkt $\mathfrak{x}_0 \in \mathfrak{G}$

$$U_0(\mathfrak{x}_0, \varrho_0(s)) = U_0(\mathfrak{x}_0, \varrho_0(s_0)) + U_0(\mathfrak{x}_0, \varrho_0(s) - \varrho_0(s_0)), \tag{6.4}$$

und es ist

$$U_0(\mathfrak{x}, \varrho_0(s_0)) + 2\varrho_0(s_0) \lg r \tag{6.5}$$

für alle $\mathfrak{x}$ gleichmäßig beschränkt. Wir müssen daher zeigen, daß auch

$$U_0(\mathfrak{x}_0, \varrho_0(s) - \varrho_0(s_0)) = U_0(\mathfrak{x}_0, \varrho_0^*(s)) \tag{6.6}$$

gleichmäßig beschränkt bleibt. Dazu stellen wir unsere Kurve in der Umgebung des Punktes $\mathfrak{y}_0$ in einem kartesischen Tangentensystem dar. Dieses hat folgende Eigenschaften:

1. $\mathfrak{y}_0$ ist Ursprung des Systems.
2. Es gibt eine Konstante $c > 0$ so, daß der im Innern der Kugel

$$x^2 + y^2 + z^2 \leqq c$$

gelegene Teil $\mathfrak{C}_c$ von $\mathfrak{C}$ in der Form

$$y = f(x); \quad z = g(x)$$

dargestellt werden kann, wobei die Funktionen f und g in $x < c$ zweimal stetig differenzierbar sind. Die Konstante c existiert für alle Punkte $\mathfrak{y}_0$ gleichmäßig.

3. Die Ableitungen f' und g' verschwinden im Ursprung.

Wir wählen nun eine feste Zahl $0<\tau<c$ und zerlegen $U_0(\mathfrak{x}_0, \varrho_0^*(s))$ im Punkte $\mathfrak{y}_0=\mathfrak{x}_0-r\sin\varphi\,\mathfrak{h}_0-r\cos\varphi\,\mathfrak{b}_0$ in der Form

$$(6.7)\qquad U_0(\mathfrak{x}_0, \varrho_0^*(s)) = \int\limits_{\mathfrak{C}_\tau} \frac{\varrho_0^*(s)\,ds}{|\mathfrak{x}_0-\mathfrak{y}|} + \int\limits_{\mathfrak{C}-\mathfrak{C}_\tau} \frac{\varrho_0^*(s)\,ds}{|\mathfrak{x}_0-\mathfrak{y}|}.$$

Die Integrale auf der rechten Seite von (6.7) sind beschränkt und können folgendermaßen abgeschätzt werden:

$$(6.8)\qquad \begin{aligned} \int\limits_{\mathfrak{C}_\tau} \frac{\varrho_0^*(s)\,ds}{|\mathfrak{x}_0-\mathfrak{y}|} &= \int \frac{\varrho_0^*(s)\,ds}{|\mathfrak{y}_0-\mathfrak{y}+r\sin\varphi\,\mathfrak{h}_0+r\cos\varphi\,\mathfrak{b}_0|} \\ &= O\left(\int\limits_{-\tau}^{\tau} \frac{|x|^\alpha\,dx}{\sqrt{x^2+(y-r\sin\varphi)^2+(z-r\cos\varphi)^2}}\right) = O(\tau^\alpha), \end{aligned}$$

$$(6.9)\qquad \left|\int\limits_{\mathfrak{C}-\mathfrak{C}_\tau} \frac{\varrho_0^*(s)\,ds}{|\mathfrak{x}_0-\mathfrak{y}|}\right| \leq \frac{A_1}{\tau}.$$

Es gilt also gleichmäßig bezüglich $\mathfrak{x}_0$ mit $0<\alpha\leqq 1$

$$(6.10)\qquad |U_0(\mathfrak{x}_0, \varrho_0^*(s))| \leqq A_2\left(\frac{1}{\tau}+\tau^\alpha\right) \leqq A_3,$$

womit Satz 4 bewiesen ist.

Zum Beweis von Satz 5 gehen wir entsprechend vor. Wir setzen

$$(6.11)\quad V^*(\mathfrak{x}) = \varrho(\mathfrak{y})\left\{\frac{2\,\mathfrak{d}(\mathfrak{y})\,(\mathfrak{x}-\mathfrak{y})}{|\mathfrak{x}-\mathfrak{y}|^2} - \varkappa(\mathfrak{y})\,(\mathfrak{h}\,\mathfrak{d})\lg|\mathfrak{x}-\mathfrak{y}| + \varkappa(\mathfrak{y})\,\frac{[(\mathfrak{x}-\mathfrak{y})\,\mathfrak{d}]\,[(\mathfrak{x}-\mathfrak{y})\,\mathfrak{h}]}{|\mathfrak{x}-\mathfrak{y}|^2}\right\}.$$

$V^*(\mathfrak{x})$ genügt für kleine r der Abschätzung

$$(6.12)\qquad \Delta V^*(\mathfrak{x}) = O\left(\frac{1}{r}\right).$$

Es existiert daher das mit V^* gebildete Integral (6.3) und unsere alten Überlegungen bleiben richtig, wenn $\varrho(s)$ zweimal und $\mathfrak{C}$ viermal stetig differenzierbar ist.

Es sei $\varrho(s)$ nun hölderstetig differenzierbar. Dann bilden wir in einem beliebigen Punkte $\mathfrak{x}_0\in\mathfrak{G}$

$$(6.13)\qquad U(\mathfrak{x}_0, \varrho(s)) = U(\mathfrak{x}_0, \varrho^1(s)) + U(\mathfrak{x}_0, \varrho^*(s)),$$

mit

$$(6.14)\qquad \begin{aligned} \varrho^1(s) &= \varrho(s_0) + (s-s_0)\,\varrho'(s_0), \\ \varrho^*(s) &= \varrho(s) - \varrho^1(s). \end{aligned}$$

Die Funktion $\varrho^*(s)$ genügt wegen der vorausgesetzten hölderstetigen Differenzierbarkeit der Abschätzung

$$(6.15)\quad |\varrho^*(s)| \leqq |s-s_0|\,|\varrho'(s_0+\delta(s-s_0)) - \varrho'(s_0)| \leqq |s-s_0|^{1+\alpha}, \qquad 0<\alpha\leqq 1.$$

Da $\varrho^1(s)$ zweimal differenzierbar ist, bleibt $U(\mathfrak{x}, \varrho^1(s)) - V(\mathfrak{x}, \varrho^1(s))$ nach obigen Überlegungen gleichmäßig beschränkt. Um auch $U(\mathfrak{x}_0, \varrho^*(s))$ abzuschätzen, benutzen wir wieder im Punkte $\mathfrak{y}_0$ das Tangentensystem und zerlegen

$$(6.16)\qquad U(\mathfrak{x}_0, \varrho^*(s)) = \int\limits_{\mathfrak{C}_\tau} \varrho^*(s)\,\mathfrak{d}\,\nabla\frac{1}{|\mathfrak{x}_0-\mathfrak{y}|}\,ds + \int\limits_{\mathfrak{C}-\mathfrak{C}_\tau} \varrho^*(s)\,\mathfrak{d}\,\nabla\frac{1}{|\mathfrak{x}_0-\mathfrak{y}|}\,ds.$$

Die Integrale der rechten Seite von (6.16) sind beschränkt und können folgendermaßen abgeschätzt werden:

$$(6.17)\quad \int\limits_{\mathfrak{C}_\tau} \varrho^*(s)\,\mathfrak{d}\,\nabla \frac{1}{|\mathfrak{x}_0-\mathfrak{y}|}\,ds = O\left(\int\limits_{-\tau}^{\tau} \frac{|x|^{1+\alpha}\,dx}{\sqrt{x^2+(y-r\sin\varphi)^2+(z-r\cos\varphi)^2}^{\,2}}\right) = O(\tau^\alpha),$$

$$(6.18)\quad \left|\int\limits_{\mathfrak{C}-\mathfrak{C}_\tau} \varrho^*(s)\,\mathfrak{d}\,\nabla \frac{1}{|\mathfrak{x}_0-\mathfrak{y}|}\,ds\right| \leqq \frac{B_1}{\tau^2}.$$

Es gilt daher gleichmäßig bezüglich $\mathfrak{x}_0$

$$(6.19)\quad |U(\mathfrak{x}_0,\varrho^*(s))| \leqq B_2\left(\frac{1}{\tau^2}+\tau^\alpha\right) \leqq B_3,$$

womit auch Satz 4 bewiesen ist.

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(Eingegangen am 25. April 1958)

Über das Neumannsche Randwertproblem für die Helmholtzsche Schwingungsgleichung

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In der Potentialtheorie sind drei Randwertprobleme des Außenraumes bekannt geworden: die Dirichletsche Aufgabe (der Wert der Lösung U ist auf dem Rand der geschlossenen, orientierbaren und glatten Fläche Ω als stetige Funktion vorgegeben), die Neumannsche Aufgabe $\left(\frac{\partial U}{\partial n} \text{ ist vorgegeben}\right)$ und die dritte Randwertaufgabe $\left(\frac{\partial U}{\partial n} + \delta U \text{ ist als stetige Funktion vorgegeben}\right)$. In allen Fällen gelingt es, die Aufgaben auf Integralgleichungen zurückzuführen und die Lösungen als Potentialfunktionen von einfachen und Dipolbelegungen darzustellen [*1*].

Eine Übertragung dieser Ergebnisse auf die Theorie der Helmholtzschen Schwingungsgleichung

$$\Delta U + k^2 U = 0 \qquad (k \neq 0, \ \operatorname{Im} k \geqq 0)$$

bietet jedoch charakteristische Schwierigkeiten und erfordert weitergehende Methoden, da die Theorie der Schwingungsgleichung nicht durch Übertragung der Ergebnisse der Potentialtheorie zu gewinnen ist. So gilt beispielsweise nicht das Maximumprinzip; auch existiert kein Analogon zur Kelvintransformation, so daß die asymptotischen Gesetze im Unendlichen nicht aus dem Verhalten im Endlichen gewonnen werden können. Andererseits interessieren gerade solche Lösungen der Schwingungsgleichung, die ein spezielles asymptotisches Verhalten zeigen, das durch die Sommerfeldsche Ausstrahlungsbedingung

$$\frac{\partial U}{\partial r} = i\,k\,U + o\left(\frac{1}{r}\right) \qquad \text{für große } r$$

beschrieben wird.

Es zeigt sich, daß die Randwertaufgaben zur Helmholtzschen Schwingungsgleichung unter der zusätzlichen Forderung dieser Ausstrahlungsbedingung eindeutig lösbar sind. Für die erste Randwertaufgabe ist das von H. WEYL [*2*] und C. MÜLLER [*3*] gezeigt worden. Die zweite und dritte Randwertaufgabe wird in dieser Arbeit durch Übertragung des Beweisgedankens von C. MÜLLER behandelt. Die beiden ersten Randwertaufgaben sind ferner von W. D. KUPRADSE [*4*] bearbeitet worden. Hier werden jedoch die Eigenwerte der auftretenden Integralgleichungen nicht diskutiert, so daß zur Lösung wesentlich umfangreichere Methoden aus der Theorie der Integralgleichungen herangezogen werden müssen.

Wir betrachten in dieser Arbeit die Schwingungsgleichung für komplexe k mit $\operatorname{Im} k \geq 0$. Unsere Betrachtungen gelten sinngemäß auch im Falle $\operatorname{Im} k < 0$, wenn wir dann für das Verhalten im Unendlichen die sogenannte Einstrahlungsbedingung

$$\frac{\partial U}{\partial r} = -i k U + o\left(\frac{1}{r}\right) \qquad \text{für große } r$$

fordern.

Im § 1 der Arbeit werden die Eigenschaften der Potentialfunktionen einfacher bzw. doppelter Flächenbelegung zusammengestellt. Der § 2 folgert die Eindeutigkeit unserer Aufgabe aus einem Ergebnis von F. RELLICH. Im § 3 werden die Lösungen als Potentialfunktionen angesetzt und für ihre Belegungsfunktionen Integralgleichungen hergeleitet. Der § 4 diskutiert die Frage der Eigenlösungen. Im letzten Paragraphen schließlich wird der Existenzbeweis nach der Methode der Kapazitätskoeffizienten gegeben.

§ 1. Vorbemerkungen

Unseren Betrachtungen liege der dreidimensionale euklidische Raum zugrunde. Er enthalte das beschränkte Gebiet G_i mit der geschlossenen Oberfläche Ω. Das Äußere dieses Gebietes bezeichnen wir mit G_a. Die Fläche Ω sei orientierbar und glatt; wo es notwendig ist, die verschiedenen Seiten von Ω zu unterscheiden, geschieht das durch angehangene Indizes $+$ (außen) und $-$ (innen). Der Normalenvektor von Ω weise nach außen. Mit p bezeichnen wir einen beliebigen Punkt im Raum, während s stets auf Ω liegen soll. $r(p, s)$ sei der Abstand zwischen p und s. Dann sind die Funktionen

$$U(p) = \int_{\Omega} \mu(s) \frac{e^{ikr}}{r} \, ds \tag{1.1}$$

und

$$V(p) = \int_{\Omega} \nu(s) \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r} \, ds \tag{1.2}$$

für alle $p \in G_a$ Lösungen der Helmholtzschen Schwingungsgleichung. Aus der Potentialtheorie übernehmen wir für diese Funktionen folgende Ergebnisse [5]:

Lemma 1. *$\mu(s)$ sei stetig. Dann ist $U(p)$ überall stetig. Für die Ableitung in Richtung der Normalen gilt bei Annäherung an Ω die Sprungrelation*

$$\frac{\partial U_+}{\partial n} = -2\pi \mu(s) + \int_{\Omega} \mu(s') \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r(s, s')} \, ds'$$

$$\frac{\partial U_-}{\partial n} = 2\pi \mu(s) + \int_{\Omega} \mu(s') \frac{\partial}{\partial n_s} \frac{e^{ikr}}{r(s, s')} \, ds'.$$

Lemma 2. *1. $\nu(s)$ sei stetig. Dann gilt*

$$V_+ = 2\pi \nu(s) + \int_{\Omega} \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s, s')} \, ds'$$

$$V_- = -2\pi \nu(s) + \int_{\Omega} \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s, s')} \, ds'$$

und es ist

$$\int_{\Omega} \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} ds' = \frac{1}{2}(V_+ + V_-)$$

hölderstetig.

2. $\nu(s)$ sei hölderstetig. Dann ist

$$\frac{\partial}{\partial s} \int_{\Omega} \nu(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} ds'$$

vorhanden und hölderstetig.

3. $\nu(s)$ sei hölderstetig differenzierbar. Dann existiert die Normalableitung von $V(p)$ und ist beim Durchgang von p durch Ω hölderstetig.

Aus Lemma 1 und Lemma 2 erhalten wir

Lemma 3. *Die stetige Funktion $\sigma(s)$ genüge einer Integralgleichung*

$$\sigma(s) = \lambda \int_{\Omega} \sigma(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(s,s')} ds'.$$

Dann existiert die Normalableitung der Funktion

$$V(p) = \int_{\Omega} \sigma(s') \frac{\partial}{\partial n_{s'}} \frac{e^{ikr}}{r(p,s')} ds'$$

und ist beim Durchgang von p durch Ω stetig.

§ 2. Der Eindeutigkeitssatz

In diesem Paragraphen beweisen wir

Satz 1. *Es gibt höchstens eine Funktion $U(p)$ mit folgenden Eigenschaften:*

1. $U(p)$ ist in G_a zweimal stetig differenzierbar.

2. $U(p)$ genügt in G_a der Differentialgleichung

$$\Delta U + k^2 U = 0 \qquad \text{mit } k \neq 0,\ \operatorname{Im} k \geqq 0.$$

3. $U(p)$ nimmt an der Außenseite von Ω die stetigen Randwerte $\gamma(s) = \frac{\partial U}{\partial n} + \delta(s) U(s)$ an. Die Funktion $\delta(s)$ ist dabei als reellwertig und stetig und im Falle Im $k \neq 0$ *kleiner oder gleich Null vorausgesetzt.*

4. $U(p)$ genügt gleichmäßig für alle Richtungen der Ausstrahlungsbedingung

$$\frac{\partial U}{\partial r} = i k U + o\left(\frac{1}{r}\right) \qquad \text{für große } r.$$

Zum Beweis von Satz 1 benötigen wir folgendes Ergebnis von Rellich [6]:

Lemma 4. *Die Funktion $U(p)$ genüge für alle $|p| \geqq c$ der Gleichung $\Delta U + k^2 U = 0$. Es sei Σ_R eine Kugel um den Ursprung mit dem Radius $R \geqq c$, und es gelte*

$$\int_{\Sigma_R} |U|^2 dF = o(1) \qquad \text{für } R \to \infty.$$

Dann verschwindet U identisch.

Es seien nun U und V zwei Lösungen unserer Randwertaufgabe. Dann erhalten wir durch $W = U - V$ eine Funktion, die auf Ω_+ den Wert

$$\frac{\partial W}{\partial n} + \delta W = 0 \tag{2.1}$$

annimmt; wir wollen zeigen, daß W identisch verschwindet.

Dazu wählen wir eine genügend große Kugel Σ mit dem Radius R, die Ω ganz im Inneren enthält, und erhalten aus der Greenschen Formel wegen

$$\int_{\Omega_+} \left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n} \right) dF = 0, \tag{2.2}$$

$$\int_G (W \Delta \overline{W} - \overline{W} \Delta W)\, dV = \int_\Sigma \left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n} \right) dF, \tag{2.3}$$

wenn G das von Ω und Σ berandete Gebiet ist. Auf Grund der Ausstrahlungsbedingung gilt nun mit $k = k_1 + i\, k_2$

$$\begin{aligned} &\left(\frac{\partial W}{\partial n} - i\, k\, W \right) \left(\frac{\partial \overline{W}}{\partial n} + i\, \bar{k}\, \overline{W} \right) \\ &\quad = \left| \frac{\partial W}{\partial n} \right|^2 + |k\, W|^2 - i\, k_1 \left(W \frac{\partial \overline{W}}{\partial n} - \overline{W} \frac{\partial W}{\partial n} \right) + 2 k_2 \operatorname{Re} \left(W \frac{\partial \overline{W}}{\partial n} \right) = o\left(\frac{1}{R^2} \right). \end{aligned} \tag{2.4}$$

Sodann ergibt sich aus der ersten Greenschen Formel

$$\int_G W \Delta \overline{W}\, dV + \int_G |\nabla W|^2\, dV = \int_\Sigma W \frac{\partial \overline{W}}{\partial n}\, dF - \int_\Omega W \frac{\partial \overline{W}}{\partial n}\, dF \tag{2.5}$$

oder

$$(-k_1^2 + k_2^2) \int_G |W|^2\, dV + \int_G |\nabla W|^2\, dV = \operatorname{Re} \int_\Sigma W \frac{\partial \overline{W}}{\partial n}\, dF - \int_\Omega W \frac{\partial \overline{W}}{\partial n}\, dF. \tag{2.6}$$

Damit folgt aus (2.3) für große R

$$\int_\Sigma \left(\left| \frac{\partial W}{\partial n} \right|^2 + |k\, W|^2 \right) dF + 2 k_2 \int_G \left(|k\, W|^2 + |\nabla W|^2 \right) dV + 2 k_2 \int_\Omega W \frac{\partial \overline{W}}{\partial n}\, dF = o(1). \tag{2.7}$$

Da nach Voraussetzung im Falle $k_2 > 0$

$$\int_\Omega W \frac{\partial \overline{W}}{\partial n}\, dF \geqq 0 \tag{2.8}$$

ist, folgt aus (2.7), daß W identisch verschwindet. Ist jedoch $k_2 = 0$, dann erhalten wir aus (2.7)

$$\int_\Sigma |W|^2\, dF = o(1). \tag{2.9}$$

Nach Lemma 4 verschwindet W auch in diesem Falle, womit Satz 1 bewiesen ist.

§ 3. Aufstellung der Integralgleichung

Nachdem wir die Frage der Eindeutigkeit der Lösungen unserer Randwertaufgaben behandelt haben, wollen wir uns nun mit dem Existenzproblem beschäftigen und die Fragestellung zunächst auf Integralgleichungen zurückführen.

Dazu betrachten wir die Funktionen

$$Q(p,s)=\frac{1}{2\pi}\,\frac{e^{ikr}}{r(p,s)}\,;\qquad K(p,s)=\frac{\partial}{\partial n_s}\,Q(p,s) \tag{3.1}$$

die für $p \neq s$ der Differentialgleichung

$$\Delta U + k^2 U = 0 \tag{3.2}$$

genügen. Die Funktionen Q und K sind Grundlösungen unseres Problems. Wir werden versuchen, aus ihnen durch Integration über Ω Lösungen der Randwertaufgaben aufzubauen, und bilden dazu mit stetigen Funktionen $\mu(s)$ und $\nu(s)$

$$U_1(p)=\int_{\Omega}\mu(s)\,Q(p,s)\,ds \tag{3.3}$$

und

$$U_2(p)=\int_{\Omega}\nu(s)\,K(p,s)\,ds\,. \tag{3.4}$$

Die Funktionen $U_1(p)$ und $U_2(p)$ genügen in G_a der Differentialgleichung (3.2) und der Ausstrahlungsbedingung.

Zur Lösung unserer Randwertaufgaben machen wir den Ansatz $U=U_1$ und erhalten nach Lemma 1

$$\frac{\partial U}{\partial n}+\delta U=\gamma(s)=-\mu(s)+\int_{\Omega}\mu(s')\,\{K(s',s)+\delta(s)\,Q(s',s)\}\,ds' \tag{3.5}$$

wofür wir symbolisch

$$\gamma=\mu-\mu K \tag{3.6}$$

schreiben. Die zugehörige transponierte Integralgleichung bezeichnen wir entsprechend mit

$$\gamma=\mu-K\mu\,. \tag{3.7}$$

Damit haben wir unsere Randwertaufgaben auf eine Integralgleichung zurückgeführt, die wir nun diskutieren wollen.

§ 4. Über die Verteilung der Eigenwerte der Integralgleichung

Die im vorigen Paragraphen aufgestellte Integralgleichung geht aus

$$\gamma=\mu+\lambda\mu K \tag{4.1}$$

für $\lambda=-1$ hervor. Zur Lösung von (3.6) untersuchen wir daher zunächst die homogene Gleichung für $\lambda=-1$.

Es sei Φ bzw. H die Gesamtheit der Lösungen der zueinander transponierten Integralgleichungen

$$\eta-K\eta=0;\qquad \varphi-\varphi K=0\,. \tag{4.2}$$

Dann bilden wir mit einem $\eta\in H$ die Funktion

$$V(p)=\int_{\Omega}\eta(s')\,\{K(p,s')+\delta(s')\,Q(p,s')\}\,ds', \tag{4.3}$$

die der Schwingungsgleichung sowie der Ausstrahlungsbedingung genügt und zeigen zunächst, daß die Randwerte von $\partial V_-/\partial n$, die nach Lemma 3 ja existieren, zu Φ gehören. Es gilt nämlich

$$V_- = -\eta + K\eta = 0 \tag{4.4}$$

woraus nach Lemma 2

$$V_+ = 2\eta \tag{4.5}$$

folgt. Wenden wir nun die Greensche Formel mit den Funktionen $V(q)$ und $Q(p, q)$ $(p \in G_a)$ auf das Gebiet G_i an, dann folgt aus (4.4)

$$0 = \int\limits_{\Omega_-} \frac{\partial V}{\partial n} Q(p, s')\, d s'. \tag{4.6}$$

Durch Differentiation entsteht aus (4.6) bei Annäherung von p an Ω_+

$$0 = -\frac{\partial V_-}{\partial n} + \int\limits_{\Omega} \frac{\partial V_-}{\partial n} K(s', s)\, d s'. \tag{4.7}$$

Multiplizieren wir (4.6) für $p = s$ mit $\delta(s)$ und addieren das Ergebnis zu (4.7), dann folgt schließlich

$$0 = -\frac{\partial V_-}{\partial n} + \int\limits_{\Omega} \frac{\partial V_-}{\partial n} \{K(s', s) + \delta(s)\, Q(s', s)\}\, d s' \tag{4.8}$$

womit die Behauptung bewiesen ist. Daraus ergibt sich

Lemma 5. *Sei* $M\eta = \frac{\partial V_-}{\partial n}$. *Dann ist die Abbildung* $\varphi = M\eta$ *umkehrbar eindeutig.*

Wegen

$$\frac{\partial V_+}{\partial n} = -2\delta\eta + \frac{\partial V_-}{\partial n} \tag{4.9}$$

folgt nämlich aus $\varphi = 0$ und (4.5)

$$\frac{\partial V_+}{\partial n} + \delta V_+ = 0. \tag{4.10}$$

Auf Grund des Eindeutigkeitssatzes erhalten wir daher

$$V \equiv 0 \qquad \text{in } G_a, \tag{4.11}$$

woraus sich wegen (4.5)

$$\eta = 0 \tag{4.12}$$

ergibt. Damit ist Lemma 5 bewiesen. Wir zeigen nun

Lemma 6. *Die Funktion* $U(p)$ *genüge in* G_a *der Differentialgleichung* $\Delta U + k^2 U = 0$ *und der Ausstrahlungsbedingung. Die Grenzwerte von* U *und* $\partial U/\partial n$ *bei Annäherung an* Ω *mögen existieren. Dann ist*

$$\operatorname{Im} \int\limits_{\Omega_+} U \frac{\partial \bar{U}}{\partial n}\, dF = -C \operatorname{Re} k \qquad \textit{mit } C \geqq 0,$$

wobei $C = 0$ *nur dann gilt, wenn* $U(p)$ *in* G_a *identisch verschwindet.*

Um Lemma 6 zu beweisen, wenden wir die Greensche Formel mit U und $\overline{U}$ auf das Gebiet G an, das von Ω und einer genügend großen Kugel Σ berandet wird. Es folgt

$$\int\limits_{\Omega_+}\left(U\frac{\partial\overline{U}}{\partial n}-\overline{U}\frac{\partial U}{\partial n}\right)dF=-\int\limits_{G}(U\,\Delta\overline{U}-\overline{U}\,\Delta U)\,dV+\int\limits_{\Sigma_R}\left(U\frac{\partial\overline{U}}{\partial n}-\overline{U}\frac{\partial U}{\partial n}\right)dF \tag{4.13}$$

oder mit $k=k_1+i\,k_2$ $(k_2\geqq 0)$

$$\operatorname{Im}\int\limits_{\Omega_+}U\frac{\partial\overline{U}}{\partial n}\,dF=-2k_1k_2\int\limits_{G}|U|^2\,dV-k_1\int\limits_{\Sigma_R}|U|^2\,dF+o(1) \tag{4.14}$$

womit der erste Teil der Behauptung von Lemma 6 bewiesen ist. Aus

$$\int\limits_{\Sigma}|U|^2\,dF=o(1)\qquad\text{für } R\to\infty \tag{4.15}$$

folgt aber nach Lemma 4, daß U in G_a identisch verschwindet.

Setzen wir nun in Lemma 6 für $U(p)$ die Funktion $V(p)$ ein, dann erhalten wir wegen

$$V_+=2\eta;\qquad\frac{\partial V_+}{\partial n}=M\eta-2\delta\eta. \tag{4.16}$$

Lemma 7. *Für alle* $\eta\in H$ *gilt*

$$\operatorname{Im}\int\limits_{\Omega}\eta\cdot\overline{M\eta}\,dF=-C\operatorname{Re}k\qquad\textit{mit }C\geqq 0$$

wobei $C=0$ *nur für* $\eta=0$ *möglich ist.*

Nach diesen Vorbereitungen beweisen wir

Satz 2. *Es sei* $\operatorname{Im}k\neq 0$. *Dann ist* $\lambda=-1$ *kein Eigenwert der Integralgleichungen*

$$0=\psi+\lambda\,\psi K$$
$$0=\varepsilon+\lambda K\varepsilon.$$

Zum Beweis betrachten wir die Funktion

$$U(p)=\int\limits_{\Omega}\varepsilon\{K(p,s')+\delta(s')\,Q(p,s')\}\,ds' \tag{4.17}$$

die auf Ω die Werte

$$U_+=\varepsilon+K\varepsilon;\qquad U_-=-\varepsilon+K\varepsilon$$
$$\frac{\partial U_+}{\partial n}=\frac{\partial U_-}{\partial n}-2\varepsilon\delta \tag{4.18}$$

annimmt. Aus (4.18) folgt

$$(1+\lambda)\,U_+-(1-\lambda)\,U_-=2(\varepsilon+\lambda K\varepsilon). \tag{4.19}$$

Es sei λ nun reeller Eigenwert. Dann verschwindet (4.19), und es gibt zwei reelle Zahlen

$$\alpha=1+\lambda;\qquad\beta=1-\lambda \tag{4.20}$$

mit

$$\alpha U_+-\beta U_-=0. \tag{4.21}$$

Durch Anwendung der Greenschen Formel auf das Gebiet G_i erhalten wir nun

$$\int\limits_{G_i} (U \Delta \bar{U} - \bar{U} \Delta U)\, dV = \int\limits_{\Omega_-} \left(U \frac{\partial \bar{U}}{\partial n} - \bar{U} \frac{\partial U}{\partial n} \right) dF \tag{4.22}$$

oder mit $k = k_1 + i k_2$

$$\operatorname{Im} \int\limits_{\Omega_-} U \frac{\partial \bar{U}}{\partial n}\, dF = 2 k_1 k_2 \int\limits_{G_i} |U|^2\, dV = k_1 k_2 D \tag{4.23}$$

mit $D \geqq 0$, wobei $D = 0$ nur dann gilt, wenn U in G_i identisch verschwindet. Zusammen mit Lemma 6 folgt daraus

$$-\operatorname{Im} \int\limits_{\Omega} \left\{ \alpha U_+ \frac{\partial \bar{U}_+}{\partial n} - \beta U_- \frac{\partial \bar{U}_-}{\partial n} \right\} dF = \alpha k_1 C + \beta k_1 k_2 D \tag{4.24}$$

woraus nach (4.18) und (4.21)

$$2\alpha \int\limits_{\Omega} \bar{\varepsilon}\, \delta U_+\, dF = \alpha k_1 C + \beta k_1 k_2 D \tag{4.25}$$

folgt.

Es sei zunächst $\operatorname{Re} k = k_1 \neq 0$ und $\lambda = -1$ Eigenwert. Dann ist $\varepsilon = \eta$, und es folgt aus (4.25) $D = 0$. Die Funktion U verschwindet also in G_i identisch, woraus insbesondere

$$\frac{\partial U_-}{\partial n} = 0 \tag{4.26}$$

folgt. Nach Lemma 5 gilt dann aber $\varepsilon = 0$; in diesem Falle ist daher $\lambda = -1$ kein Eigenwert.

Sei nun $Re\, k = k_1 = 0$. Dann folgt durch Anwendung der ersten Greenschen Formel

$$\int\limits_{\Omega_-} U \frac{\partial \bar{U}}{\partial n}\, dF = k_2^2 \int\limits_{G_i} |U|^2\, dV + \int\limits_{G_i} |\nabla U|^2\, dV \tag{4.27}$$

oder

$$\int\limits_{\Omega_-} U \frac{\partial \bar{U}}{\partial n}\, dF = D_1 \geqq 0. \tag{4.28}$$

Andererseits gilt

$$\int\limits_{\Omega_+} U \frac{\partial \bar{U}}{\partial n}\, dF = -k_2^2 \int\limits_{G_a} |U|^2\, dV - \int\limits_{G_a} |\nabla U|^2\, dV - k_2 \int\limits_{\Sigma_R} |U|^2\, dF + o(1) \tag{4.29}$$

oder

$$\int\limits_{\Omega_+} U \frac{\partial \bar{U}}{\partial n}\, dF = -C_1 \qquad \text{mit } C_1 \geqq 0. \tag{4.30}$$

Wir erhalten daher jetzt an Stelle von (4.24)

$$-\int\limits_{\Omega} \left\{ \alpha U_+ \frac{\partial \bar{U}_+}{\partial n} - \beta U_- \frac{\partial \bar{U}_-}{\partial n} \right\} dF = \alpha C_1 + \beta D_1 \tag{4.31}$$

oder

$$2\alpha \int\limits_{\Omega} \bar{\varepsilon}\, \delta U_+\, dF = \alpha C_1 + \beta D_1. \tag{4.32}$$

Angenommen $\lambda = -1$ wäre Eigenwert. Dann verschwindet D_1 und nach (4.27) U in G_i, was wieder zum Widerspruch führt. Damit ist Satz 2 bewiesen.

In Satz 2 haben wir eine Aussage über die Verteilung der Eigenwerte unserer Integralgleichung im Falle $\operatorname{Im} k \neq 0$ erhalten. Wir beschäftigen uns nun mit reellen k und beweisen

Satz 3. *Sei $k \neq 0$ reell. Dann sind die reellen Eigenwerte der Integralgleichungen*

$$0 = \psi + \lambda \psi K$$

$$0 = \varepsilon + \lambda K \varepsilon$$

einfach[1] *und entweder gleich* $+1$ *oder* -1.

Zum Beweis betrachten wir die Funktion

$$U(p) = \int_{\Omega} \psi\, Q(p, s')\, ds' \tag{4.33}$$

die auf Ω die Werte

$$\begin{aligned} \frac{\partial U_+}{\partial n} &= -\psi + \psi K_1; & \frac{\partial U_-}{\partial n} &= \psi + \psi K_1 \\ U_+ &= U_-; & \psi K_1 &= \int_{\Omega} \psi(s')\, K(s', s)\, ds' \end{aligned} \tag{4.34}$$

annimmt. Aus (4.34) folgt

$$-(1-\lambda)\frac{\partial U_+}{\partial n} + (1+\lambda)\frac{\partial U_-}{\partial n} = 2(\psi + \lambda \psi K) - 2\lambda\, \delta U. \tag{4.35}$$

Es sei λ nun reeller Eigenwert. Dann gibt es zwei reelle Zahlen

$$\alpha = 1 + \lambda; \quad \beta = 1 - \lambda \tag{4.36}$$

mit

$$\alpha \frac{\partial U_-}{\partial n} - \beta \frac{\partial U_+}{\partial n} = -2\lambda\, \delta U. \tag{4.37}$$

Aus (4.37) folgt analog (4.21) ff.

$$\operatorname{Im} \int_{\Omega} U \left\{ \alpha \frac{\partial \bar{U}_-}{\partial n} - \beta \frac{\partial \bar{U}_+}{\partial n} \right\} dF = \beta k_1 C \tag{4.38}$$

oder

$$0 = \beta k_1 C = (1 - \lambda) k_1 C. \tag{4.39}$$

Angenommen U ist in G_a von Null verschieden. Dann folgt $\lambda = 1$. Wenn U jedoch in G_a identisch verschwindet, ergibt sich aus (4.37)

$$(1 + \lambda) \frac{\partial U_-}{\partial n} = 0, \tag{4.40}$$

das heißt, es ist $\lambda = -1$ oder U verschwindet überall identisch; dann verschwindet aber auch ψ. $\lambda = \pm 1$ sind also die einzigen reellen Eigenwerte. Wir wollen zeigen, daß sie auch einfach sind.

Dazu nehmen wir an, λ wäre mehrfacher Eigenwert. Dann gibt es Funktionen ψ^α und ε^α, die nicht identisch verschwinden mit [7]

$$\begin{aligned} \psi^1 + \lambda \psi^1 K &= 0; & \psi^2 + \lambda \psi^2 K &= \psi^1 K \\ \varepsilon^1 + \lambda K \varepsilon^1 &= 0; & \varepsilon^2 + \lambda K \varepsilon^2 &= K \varepsilon^1. \end{aligned} \tag{4.41}$$

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[1] Definiert in Gl. (4.41)

Wir wollen (4.41) für $\lambda = \pm 1$ zum Widerspruch führen. Dazu wählen wir die Funktionen

$$U_{1\alpha} = \int_{\Omega} \psi^{\alpha}\, Q(p, s')\, ds' \tag{4.42}$$

und

$$U_{2\alpha} = \int_{\Omega} \varepsilon^{\alpha} \{K(p, s') + \delta(s')\, Q(p, s')\}\, ds'. \tag{4.43}$$

Nach (4.19) und (4.35) folgt

$$\begin{aligned} -(1-\lambda)\frac{\partial U_{11+}}{\partial n} + (1+\lambda)\frac{\partial U_{11^-}}{\partial n} &= -2\lambda\,\delta U_{11} \\ -(1-\lambda)\frac{\partial U_{12+}}{\partial n} + (1+\lambda)\frac{\partial U_{12-}}{\partial n} &= \frac{\partial U_{11^+}}{\partial n} + \frac{\partial U_{11^-}}{\partial n} - 2\delta(\lambda U_{12} - U_{11}) \end{aligned} \tag{4.44}$$

und

$$\begin{aligned} (1+\lambda)\, U_{21^+} - (1-\lambda)\, U_{21^-} &= 0 \\ (1+\lambda)\, U_{22^+} - (1-\lambda)\, U_{22^-} &= 2K\,\varepsilon^1 = U_{21^+} + U_{21^-}. \end{aligned} \tag{4.45}$$

Wählen wir zunächst $\lambda = 1$. Dann folgt aus (4.44)

$$\begin{aligned} \frac{\partial U_{11-}}{\partial n} &= -\delta U_{11} \\ 2\frac{\partial U_{12^-}}{\partial n} &= \frac{\partial U_{11^+}}{\partial n} - 2\delta U_{12} + \delta U_{11}. \end{aligned} \tag{4.46}$$

Multiplizieren wir die erste dieser Gleichungen mit $2\overline{U}_{12}$, gehen wir in der zweiten zur konjugiert komplexen über und multiplizieren sie dann mit $-U_{11^+}$, so folgt durch Addition

$$2\left\{\frac{\partial U_{11-}}{\partial n}\overline{U}_{12} - \frac{\partial \overline{U}_{12^-}}{\partial n} U_{11}\right\} = -\frac{\partial \overline{U}_{11^+}}{\partial n} U_{11} - \delta U_{11}\overline{U}_{11}. \tag{4.47}$$

Integrieren wir (4.47) über Ω, dann verschwindet die linke Seite des Integrals nach der Greenschen Formel, und wir erhalten

$$0 = \operatorname{Im} \int_{\Omega_+} U_{11} \frac{\partial \overline{U}_{11}}{\partial n}\, dF. \tag{4.48}$$

Nach Lemma 6 ist das nur möglich, wenn U_{11} in G_a identisch verschwindet. Dann folgt aber aus (4.46)

$$\frac{\partial U_{11^+}}{\partial n} = \frac{\partial U_{11^-}}{\partial n} = 0. \tag{4.49}$$

Nach Lemma 1 ist daher entgegen der Annahme $\psi_1 = 0$.

Sei nun $\lambda = -1$. Dann erhalten wir aus (4.45)

$$U_{21^-} = 0; \qquad -2U_{22^-} = U_{21^+} \tag{4.50}$$

woraus jetzt

$$2\left\{U_{21^-}\frac{\partial \overline{U}_{22-}}{\partial n} - \overline{U}_{22^-}\frac{\partial U_{21^-}}{\partial n}\right\} = \overline{U}_{21^+}\frac{\partial U_{21-}}{\partial n} = \overline{U}_{21^+}\frac{\partial U_{21^+}}{\partial n} + 2\varepsilon^1\delta\,\overline{U}_{21^+} \tag{4.51}$$

folgt. Nach (4.18) gilt aber

$$U_{21^+} = 2\varepsilon^1, \tag{4.52}$$

so daß aus (4.51) nach Integration über Ω

$$0 = \operatorname{Im} \int_{\Omega_+} U_{21} \frac{\partial \overline{U}_{21}}{\partial n}\, dF \tag{4.53}$$

folgt. Gl. (4.53) ist wieder nur dann möglich, wenn U_{21} in G_a identisch verschwindet. Dann gilt aber nach (4.50)

$$U_{21^+} - U_{21^-} = 2\varepsilon^1 = 0, \tag{4.54}$$

so daß ε_1 entgegen der Annahme verschwindet. Damit ist auch Satz 3 bewiesen.

§ 5. Auflösung der Integralgleichung und Beweis des Existenzsatzes

Wir haben im letzten Paragraphen Aussagen über die Verteilung der Eigenwerte unserer Integralgleichung gewonnen und sind nach diesen Vorbereitungen in der Lage, den Existenzsatz zu unserem Randwertproblem zu beweisen. Wir zeigen

Satz 4. *Es gibt eine Funktion $U(p)$, die auf Ω die Randwerte $\gamma = \frac{\partial U}{\partial n} + \delta U$ annimmt und im übrigen den Voraussetzungen von Satz 1 genügt.*

Im § 3 haben wir diese Randwertaufgabe auf die Lösung der Integralgleichung

$$\gamma = \mu - \mu K \tag{5.1}$$

zurückgeführt. Wie wir in Satz 2 sahen, besitzt sie für $\mathrm{Im}(k) \neq 0$ keine Eigenlösungen und ist in diesem Falle also stets lösbar [7]; wir brauchen uns daher nur noch mit dem Fall reeller k $(k = k_1)$ zu beschäftigen.

Wir setzen also fortan k reell $(k = k_1)$ voraus und müssen in diesem Fall die Eigenlösungen von (5.1) diskutieren. Dazu zeigen wir

Lemma 8. *Es sei k reell. Dann gehört mit φ auch $\overline{\varphi}$ zu Φ.*

Der Beweis folgt analog zu (4.4—8), wenn wir statt V die Funktion $\overline{V}$ verwenden.

Es sei nun η^i eine Basis von H. Dann erhalten wir nach Lemma 5 durch $\varphi^i = M\eta^i$ eine Basis von Φ und beweisen

Lemma 9. *Die Matrix* $$C^{ik} = \int_\Omega \varphi^i \eta^k \, dF = (\varphi^i, \eta^k)$$
ist symmetrisch und nicht entartet.

Denn wäre $\det C^{ik} = 0$, so gäbe es ein nichttriviales $\eta^* \in H$, derart, daß für alle $\varphi \in \Phi$

$$(\varphi, \eta^*) = 0 \tag{5.2}$$

gälte. Da nach Lemma 8 auch $\overline{M\eta^*}$ zu Φ gehört, folgte

$$(\eta^*, \overline{M\eta^*}) = 0. \tag{5.3}$$

Das ist nach Lemma 7 nicht möglich. Um die Symmetrie der Matrix C^{ik} zu zeigen, setzen wir

$$V^i(p) = \int_\Omega \eta^i \{K(p,s) + \delta(s)\, Q(p,s)\}\, ds. \tag{5.4}$$

An der Außenseite von Ω gilt dann

$$V^i = 2\eta^i; \tag{5.5}$$

$$\frac{\partial V}{\partial n} = M\eta^i - 2\delta\eta^i, \tag{5.6}$$

und durch Anwendung der Greenschen Formel folgt

$$\begin{aligned}(5.7)\qquad \int_{\Sigma_R}\left(V^i\frac{\partial}{\partial n}V^k - V^k\frac{\partial}{\partial n}V^i\right)dF &= \int_{\Omega_+}\left(V^i\frac{\partial}{\partial n}V^k - V^k\frac{\partial}{\partial n}V^i\right)dF\\ &= 2(\eta^i, M\eta^k) - 2(\eta^k, M\eta^i) = 2(C^{ki} - C^{ik}).\end{aligned}$$

Wegen der Ausstrahlungsbedingung

$$(5.8)\qquad \frac{\partial}{\partial R}V^i = i\,k\,V^i + o\left(\frac{1}{R}\right)$$

und

$$(5.9)\qquad V^i = O\left(\frac{1}{R}\right)$$

gilt jedoch für große R

$$(5.10)\qquad \int_{\Sigma_R}\left(V^i\frac{\partial}{\partial n}V^k - V^k\frac{\partial}{\partial n}V^i\right)dF = o(1).$$

Damit ergibt sich aus (5.7)

$$(5.11)\qquad C^{ki} - C^{ik} = o(1) \quad \text{für} \quad R\to\infty$$

und Lemma 9 ist bewiesen.

Diese Vorbereitungen ermöglichen es uns, Satz 4 auch im Falle reeller k zu beweisen. Dazu bestimmen wir zunächst zu γ die durch die Bedingungen

$$(5.12)\qquad (\eta, \gamma - \eta^*) = 0 \quad \text{für} \quad \text{alle } \eta\in H$$

eindeutig festgelegte Funktion $\eta^*\in H$. Da die Matrix C^{ik} nicht entartet ist, können wir zu η^* eindeutig ein $\varphi^*\in\Phi$ mit

$$(5.13)\qquad (\eta, \eta^* - \varphi^*) = 0 \quad \text{für} \quad \text{alle } \eta\in H$$

bestimmen. Für dieses φ^* gilt dann auch

$$(5.14)\qquad (\eta, \gamma - \varphi^*) = 0 \quad \text{für} \quad \text{alle } \eta\in H.$$

Die Integralgleichung

$$(5.15)\qquad \mu - \mu K = \gamma - \varphi^*;\quad 0 = (\mu, \varphi) \quad \text{für} \quad \text{alle } \varphi\in\Phi$$

ist daher eindeutig nach μ auflösbar. Bestimmen wir noch η' aus der Gleichung

$$(5.16)\qquad \varphi^* = M\eta',$$

dann erhalten wir die gesuchte Lösung unserer Randwertaufgabe in der Form

$$(5.17)\qquad U(p) = \int_\Omega \mu(s)\,Q(p,s)\,ds + \int_\Omega \eta'(s)\,\{K(p,s) + \delta(s)\,Q(p,s)\}\,ds.$$

Es gilt nämlich an der Außenseite von Ω wegen (4.16)

$$(5.18)\qquad \delta U + \frac{\partial U}{\partial n} = -\mu + \mu K + M\eta' = \gamma,$$

womit die zweite und die dritte Randwertaufgabe des Außenraumes für die Helmholtzsche Schwingungsgleichung gelöst sind.

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(Eingegangen am 25. April 1958)

Die Anwendung des Operatorenkalküls von Jan Mikusiński auf lineare Integralgleichungen vom Faltungstypus

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1. Orientierung und Problemstellung

Die Operatorenrechnung, die Jan Mikusiński [*10*], [*11*] eingeführt hat, erlaubt Anfangs- wie Randwertprobleme bei gewöhnlichen und partiellen Differentialgleichungen mit konstanten Koeffizienten ganz einfach kalkülmäßig oder symbolisch zu lösen. Nur solche Differentialgleichungen werden betrachtet, deren Lösungen in einer geschlossenen Form zu erhalten sind. Bisher sind verschiedene Methoden bekannt, um diese Gleichungen zu lösen, wie z.B. die Methode von Heaviside, die der Integraltransformationen (Laplace, Fourier, Hankel, Mellin usw. [*3*], [*23*], [*20*], [*21*], [*25*]), die Methode, die sich auf die Theorie der analytischen Funktionalen von L. Fantappiè stützt, und in neuester Zeit der Distributionenkalkül von L. Schwartz ([*11*a], [*17*], [*19*]).

Bekanntlich läßt sich die Operatorenrechnung im elementaren Fall der gewöhnlichen Differentialgleichungen auf algebraischem Wege (ohne Laplace-Transformationen) begründen ([*3*] Bd. II). Mit anderen Worten: in diesem Fall läßt sich der Heaviside-Kalkül in einem gewissen Sinne mathematisch rechtfertigen. Dagegen war im transzendenten Fall der partiellen Differentialgleichungen bis vor kurzem die Laplace-Transformation das einzige Mittel, um dieses Problem exakt *kalkülmäßig* zu lösen. Nun ist einer der vielen Vorteile des Mikusiński-Kalküls die exakte selbständige Fundierung des letzten Problems auf algebraischem oder symbolischem Wege.

Auch ist die Theorie von Mikusiński allgemeiner und einfacher als die auf der Laplace-Transformation beruhenden Darstellung, da z.B. das Verhalten der gesuchten Funktion im Unendlichen bei der Laplace-Transformation eine Beschränkung auferlegt, die zu gewissen Schwierigkeiten bei den Eindeutigkeitssätzen führt. Sogar wird die Kenntnis der Funktionentheorie nicht vorausgesetzt.

Es ist zu erwähnen, daß der Verfasser [*2*] dieses Problems mittels der Laplace-Transformationen vom Standpunkt der Funktionalanalysis und insbesondere der Halbgruppentheorie betrachtet hat.

Die vorliegenden Resultate wurden am 10. April 1958 in einem Vortrag anläßlich der Gamm-Tagung an der Universität des Saarlandes mitgeteilt.

Im Vergleich mit der Theorie von FANTAPPIÈ[1] oder SCHWARTZ[2] braucht sie nicht den großen topologischen und funktionalanalytischen Aufwand. Die Erweiterung des Begriffs der natürlichen Zahlen zu den rationalen und den reellen Zahlen ist ein Modell für die Methode von MIKUSIŃSKI. Kurz gesagt: die stetigen Funktionen einer reellen Variablen t, $t \geqq 0$, bilden bei der gewöhnlichen Addition und der Faltung als Multiplikation eine kommutative Algebra, die zu einem Quotientenkörper $\boldsymbol{K}$ erweitert werden kann (da diese Algebra Null-Teiler frei ist), dessen Elemente dann Operatoren genannt werden. Dieser Operatorenkörper $\boldsymbol{K}$ enthält die komplexen Zahlen, die stetigen Funktionen, den Differential- und Integraloperator. Der Dirac-Delta-Operator ist das Einheitselement dieses Körpers $\boldsymbol{K}$.

Um über die Algebra der Operatoren hinaus zu einem Infinitesimalkalkül vorzustoßen, wird zunächst in der kommutativen Algebra ein Konvergenzbegriff eingeführt und dieser dann in geeigneter Weise auf Folgen von Operatoren übertragen.

In dieser Arbeit wird die Methode von MIKUSIŃSKI auf lineare Volterrasche Integralgleichungen vom Faltungstypus von erster und zweiter Art angewandt. Auch werden Systeme von linearen Integralgleichungen sowie Integrodifferentialgleichungen und singuläre Integralgleichungen zusammen mit verschiedenen Beispielen betrachtet. Über nicht-lineare Integralgleichungen läßt sich auch etwas aussagen. Integralgleichungen, die „verallgemeinerte" Funktionen wie z.B. die Diracsche Deltafunktion enthalten, lassen sich mittels dieser Theorie auflösen. Diese kommen in der Theorie der elektrischen Netze (z.B. der Kurzschlußstrom) und des Balkens vor. Zum Beispiel lassen sich die Einzelkräfte und Einzelmomente in der Theorie des Balkens nicht durch gewöhnliche Funktionen wiedergeben, können jedoch durch Operatoren dargestellt werden. In § 5 wird ein anderes Beispiel einer Integralgleichung angegeben, welches sich auch nicht mittels der Laplace-Transformation lösen läßt, weil das entsprechende Laplace-Integral divergiert.

Diese Arbeit ist absichtlich einfach abgefaßt, so wie es in der Operatorenrechnung üblich und wünschenswert ist.

2. Vorkenntnisse aus der Operatorentheorie von Mikusiński

Es sei $\boldsymbol{C}$ die Klasse derjenigen Funktionen, die im Intervall $0 \leqq t < +\infty$ definiert und stetig sind. Wir unterscheiden zwischen der *Funktion* $f = \{f(t)\}$ und dem *Wert* $f(t)$ der Funktion im Punkte t (eine Zahl). Wir definieren die

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[1] Eine zusammenfassende Einführung in diese Theorie mit Anwendungen auf Differentialgleichungen findet man in F. PELLEGRINO [*12*] oder [*13*]. Jedoch ist zu bemerken, daß die topologische Grundlegung der Theorie von L. FANTAPPIÈ, die in verschiedener Hinsicht recht unbefriedigend war, erst in jüngster Zeit in der Habilitationsschrift von H. G. TILLMANN [*24*] dargelegt wurde. Für den Zusammenhang zwischen dieser Theorie und der auf der Laplace-Transformation beruhenden Methode, siehe auch [*1*].

[2] J. KOREVAAR [*7*] gibt vom Standpunkt der angewandten Mathematik aus eine Einführung in diese Theorie, die sehr rasch zu den Anwendungen führt. Siehe auch J. MIKUSIŃSKI und R. SIKORSKI [*11*a], sowie E SILVA [*19*]. Eine algebraische Begründung der Distributionentheorie hat H. KÖNIG [*6*] angegeben.

Addition und das *Produkt* (Faltungsprodukt) zweier Funktionen f und g durch

$$(2.1) \qquad f+g=\{f(t)+g(t)\} \quad \text{bzw.} \quad fg=\{f(t)\}\{g(t)\}=\left\{\int_0^t f(t-u)\,g(u)\,du\right\};$$

das Skalarprodukt durch $\alpha\{f(t)\}=\{\alpha f(t)\}$ wo α eine Zahl (reell oder komplex) ist. Dies führt zu einer kommutativen Algebra (ohne Einheitselement), die nach einem Satz von Titchmarsh ([*23*] S. 328 oder [*11*] S. 9—17) Null-Teiler frei ist. In anderen Worten ist eine Erweiterung zu einer kommutativen *Divisions*algebra, der sogenannte Operatorenkörper $\boldsymbol{K}$, möglich. Die Elemente dieses Körpers, die Operatoren, sind Brüche f/g für $f, g \in \boldsymbol{C}$, $g \neq \{0\}$ (natürlich versteht man hierbei nicht die gewöhnliche Division, sondern die zur Faltung inverse Operation) und die Rechenregeln entsprechen denen der rationalen Zahlen. Dieser Körper $\boldsymbol{K}$ enthält die Funktionen $f \in \boldsymbol{C}$ (weil sie in der Gestalt fg/g, wo $g \neq \{0\}$ eine beliebige Funktion aus $\boldsymbol{C}$ ist, darstellbar sind) während nicht jeder Operator eine Funktion darstellt. So ist ein Operator eine Verallgemeinerung des Begriffs einer Funktion. Auch enthält $\boldsymbol{K}$ die komplexen Zahlen (die in der Gestalt $\alpha=\alpha g/g$, $g \neq \{0\}$ darstellbar sind). So wird der Begriff der komplexen Zahl dem Begriff des Operators untergeordnet. Die Funktion $l=\{1\}$ mit der Eigenschaft $l\{f(t)\}=\left\{\int_0^t f(u)\,du\right\}$ wird Integrationsoperator genannt. Im allgemeinen gilt für $n=1, 2, \ldots$

$$l^n=\left\{\frac{t^{n-1}}{(n-1)!}\right\},$$

und die Faltung von $\{f(t)\}$ mit l^n bedeutet n-malige Integration von $\{f(t)\}$ über das Integral von 0 bis t. Wieder gehört l^n zu $\boldsymbol{K}$ (sogar zu $\boldsymbol{C}$). Bedeutet $\boldsymbol{L}$ die Klasse derjenigen Funktionen $\{f(t)\}$ die lokal summierbar sind (im Sinne von Lebesgue[3]), d.h. die auf jedem Intervall $0 \leqq t \leqq t_0$ summierbar sind, so kann auch die Klasse $\boldsymbol{L}$ in den Körper $\boldsymbol{K}$ eingebettet werden (in dem man f mit $\frac{1}{l}\left\{\int_0^t f(u)\,du\right\}$ als Quotient zweier Funktionen aus $\boldsymbol{C}$, identifiziert). Hierbei ist 1 das Einheitselement l von $\boldsymbol{K}$, welches der Diracschen „Deltafunktion" δ entspricht[4]. Auch enthält $\boldsymbol{K}$ den Differentialoperator $s=1/l$.

Ist $\{f(t)\}$ eine total-stetige[5] Funktion für $t \geqq 0$, so gilt ([*10*] S. 47, [*11*] S. 103)

$$(2.2) \qquad sf=f'+f(0),$$

wobei $f(0)$ der Wert der Funktion f im Punkte $t=0$ ist (ein Zahlenoperator) und $f'=\{f'(t)\}$ die Ableitung von f ist. Ist $f(0)=0$, so bedeutet $sf=f'$ einfache Differentiation. Aber das Produkt sf hat stets einen Sinn (sogar wenn f nicht differenzierbar ist). Allgemeiner, ist die $(n-1)$-te Ableitung $f^{(n-1)}$ total-stetig und $f^{(k)} = \{f^{(k)}(t)\}$ so gilt

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[3] Der Einfachheit halber, könnten wir annehmen, daß $\{f(t)\}$ nur endlich viele Unstetigkeitsstellen in jedem endlichen Intervall besitzt und dort *absolut* integrierbar ist.

[4] Es ist zu bemerken, daß vor kurzem C. Schmieden und D. Laugwitz [*16*] auch eine Theorie dargelegt haben, die u.a. zu einer Erweiterung des Funktionsbegriffes führt und so auch die Delta-Funktion enthält.

[5] Einfachheitshalber könnten wir die stärkere Voraussetzung $f \in \boldsymbol{C}$, wobei die Ableitung f' zur Klasse gehört, die in Fußnote 3 erklärt ist, benutzen.

$$s^n f = f^{(n)} + f^{(n-1)}(0) + s f^{(n-2)}(0) + \cdots + s^{n-1} f(0) \tag{2.3}$$

wo $f^{(k)}(0)$ den Wert von $f^{(k)}$ im Punkte $t=0$ bedeutet.

Als Anwendung dieser Überlegungen, erhält man z.B.

$$\{e^{\alpha t}\} = \frac{1}{s-\alpha}, \qquad \{\sin \alpha t\} = \left\{\frac{\alpha}{s^2+\alpha^2}\right\}. \tag{2.4}$$

Ist $f = \{f(\lambda)\} \in \boldsymbol{L}$, so hat Cz. Ryll-Nardzewski ([*10*] S. 62, [*11*] S. 322) gezeigt, daß

$$\int_0^\infty e^{-s\lambda} f(\lambda)\, d\lambda = \{f(t)\} \tag{2.5}$$

gilt, wobei dieses Integral als ein abstraktes Integral (ein Operator) im Sinne von Mikusiński ([*10*] S. 61, [*11*] S. 306) zu verstehen ist (hier bedeutet $e^{-s\lambda}$, $\lambda > 0$, den Verschiebungsoperator). Ist $\{f(t)\} \in \boldsymbol{L}$ und betrachtet man s als eine komplexe Veränderliche (nicht als den Differentialoperator) so kann man das obige Integral als ein gewöhnliches Integral auffassen und als eine Transformation die jeder Funktion $f \in \boldsymbol{L}$ der reellen Veränderlichen $t \geq 0$ (für welches das Integral *konvergiert*) eine analytische Funktion der Veränderlichen s zuordnet. So haben wir eine formale Beziehung zwischen der Laplace-Transformation und den Operatoren. Aber beide Methoden sind nicht äquivalent, da die Laplace-Transformation nur anwendbar ist, falls das Integral (2.5) konvergent ist. Dagegen hat das Integral (2.5) im Sinne von Mikusiński immer einen Sinn falls nur $f \in \boldsymbol{L}$. So ist es auch möglich die bekannten Tabellen zur Laplace-Transformation zu verwenden.

Eine Folge von Funktionen $f_1, f_2, \ldots \in \boldsymbol{C}$, ist ***C**-konvergent* gegen $f \in \boldsymbol{C}$, oder $\boldsymbol{C}\text{-}\lim\limits_{n\to\infty} f_n = f$ genau dann, wenn $f_n(t) \to f(t)$, $n \to \infty$, gleichmäßig in *jedem* abgeschlossenen Intervall $0 \leq t \leq t_0$. Eine Folge von Operatoren $f_1, f_2, \ldots \in \boldsymbol{K}$ ist ***K**-konvergent* gegen $f \in \boldsymbol{K}$, $\boldsymbol{K}\text{-}\lim\limits_{n\to\infty} f_n = f$, dann und nur dann wenn es ein Element $q \in \boldsymbol{C}$, $q \neq \{0\}$ gibt, so daß $q f_n$ $\boldsymbol{C}$-konvergent gegen $q f$ für $n \to \infty$ ist. Zum Beispiel gilt

$$\boldsymbol{K}\text{-}\lim_{n\to\infty} \left\{\frac{n}{\sqrt{\pi}} e^{-n^2 t^2}\right\} = 1,$$

welches eine Approximation des Einheitsoperators von $\boldsymbol{K}$ durch eine Folge gewöhnlicher Funktionen aus $\boldsymbol{C}$ darstellt[6]. Man sagt eine Reihe $\sum\limits_{n=1}^{\infty} f_n$ sei gegen h konvergent, falls $\lim\limits_{n\to\infty} (f_1 + f_2 + \cdots + f_n) = h$ gilt (wobei beide Konvergenzarten möglich sind).

3. Drei Sätze über Potenzreihen von Operatoren

Bekanntlich gilt ([*10*] S. 64, [*11*] S. 155, [*5*] S. 39)

Satz 3.1. *Ist* $f \in \boldsymbol{C}$, *so folgt*

$$\sum_{n=1}^{\infty} f^n = \frac{f}{1-f}, \qquad (\textit{wo } f^1 = f,\ f^2 = ff,\ \textit{usw.}). \tag{3.1}$$

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[6] Eine entsprechende Eigenschaft gilt in der Distributionentheorie, z.B. [*18*], Livre II S. 29. Der Zusammenhang zwischen dieser Theorie und der von Mikusiński ist z.B. in Temple [*22*] betrachtet.

Vollständigkeitshalber skizzieren wir den Beweis. Wähle ein beliebiges festes $t_0 > 0$. Da $f \in \boldsymbol{C}$ ist, existiert ein M mit $|f(t)| \leqq M$ für $0 \leqq t \leqq t_0$. Wir haben unter der Bezeichnung $f^n = \{f^n(t)\}$, $n = 1, 2, 3, \ldots$

$$f^1(t) = f(t), \qquad f^{n+1}(t) = \int_0^t f(t-u)\, f^n(u)\, du$$

und durch Induktion gilt

$$|f^n(t)| \leqq \frac{M^n t^{n-1}}{(n-1)!}, \qquad n = 1, 2, \ldots, \quad 0 \leqq t \leqq t_0.$$

Hieraus folgt die gleichmäßige Konvergenz der Reihe $\sum\limits_{n=1}^{\infty} f^n(t)$ in $0 \leqq t \leqq t_0$ die dort eine stetige Funktion darstellt. Da aber t_0 beliebig ist, ist die Reihe $\sum\limits_{n=1}^{\infty} f^n$ im Sinne der $\boldsymbol{C}$-Konvergenz zu einem Grenzwert $h \in \boldsymbol{C}$ konvergent. Um h zu finden setzen wir $\sum\limits_{n=1}^{\infty} f^n = h$. So gilt

$$(1-f)\, h = (1-f) \sum_{n=1}^{\infty} f^n = \sum_{n=1}^{\infty} f^n - \sum_{n=1}^{\infty} f^{n+1} = f,$$

woraus $h = f/(1-f)$ folgt, w. z. b. w.

Im Falle, daß f quadratisch integrierbar ist (oder im Sinne von Lebesgue daß $f \in \boldsymbol{L}^2$ ist) beweisen wir nun

Satz 3.2. *Gilt* $f \in \boldsymbol{L}^2$, *so folgt*

$$h = \sum_{n=1}^{\infty} f^n = \frac{f}{1-f} \quad \textit{mit} \quad h \in \boldsymbol{L}^2. \tag{3.2}$$

Beweis. Wir nehmen irgendein $t_0 > 0$ und halten es fest. Da $f \in \boldsymbol{L}^2$ ist, setzen wir

$$\int_0^{t_0} |f(u)|\, du = M_1, \qquad \int_0^{t_0} |f(u)|^2\, du = M_2.$$

Die folgenden Ungleichungen gelten alle auf dem Intervall $0 \leqq t \leqq t_0$. Zunächst gilt $|l\, f(t)| \leqq M_1$. Nach der Cauchy-Schwarzschen Ungleichung gilt

$$|f^2(t)| = \left| \int_0^t f(t-u)\, f(u)\, du \right| \leqq \left[\int_0^t |f(t-u)|^2\, du \int_0^t |f(u)|^2\, du \right]^{\frac{1}{2}} \leqq M_2,$$

woraus $|l\, f^2(t)| \leqq M_2\, t/1!$ folgt.

Auch gilt

$$|f^3(t)| = \left| \int_0^t f(t-u)\, f^2(u)\, du \right| \leqq M_2 \int_0^t |f(t-u)|\, du \leqq M_2 M_1,$$

und so ist $|l\, f^3(t)| \leqq M_2 M_1\, t/1!$.

Ferner ist

$$|f^4(t)| = |f^2 f^2(t)| \leqq M_2^2 \int_0^t du = M_2^2\, t/1!,$$

und so gilt $|l\, f^4(t)| \leqq M_2^2 \int\limits_0^t u\, du = M_2^2\, t^2/2!$.

Auch gilt

$$|f^5(t)| = |f^3 f^2(t)| \leqq M_2^2 M_1 \int\limits_0^t du = M_2^2 M_1 t/1!,$$

woraus $|l f^5(t)| \leqq M_2^2 M_1 t^2/2!$ folgt.

Im allgemeinen (durch Induktion) gilt

$$|f^{2n}(t)| \leqq M_2^n t^{n-1}/(n-1)!, \qquad |f^{2n+1}(t)| \leqq M_2^n M_1 t^{n-1}/(n-1)!$$

mit

$$|l f^{2n}(t)| \leqq M_2^n t^n/n!, \qquad |l f^{2n+1}(t)| \leqq M_2^n M_1 t^n/n!.$$

Nun ist $\sum\limits_{n=1}^{\infty} l f^n(t)$ eine (gewöhnliche) unendliche Reihe von stetigen Funktionen die im Intervall $0 \leqq t \leqq t_0$ durch die Reihe

$$\sum_{n=0}^{\infty} M_2^n (1+M_1) \frac{t_0^n}{n!} = (1+M_1)\, e^{M_2 t_0}$$

majorisiert wird, und also gleichmäßig konvergent in $0 \leqq t \leqq t_0$ ist. Also stellt die Reihe $\sum\limits_{n=1}^{\infty} l f^n(t)$ eine stetige Funktion dar (alle Glieder sind stetig). Folglich existiert $\boldsymbol{C}$-$l \sum\limits_{n=1}^{\infty} f^n$ und daher $\boldsymbol{K}$-$\sum\limits_{n=1}^{\infty} f^n$. Um den $\boldsymbol{K}$-Grenzwert zu finden, braucht man dieselbe Methode wie im vorhergehenden Beweis. Auch ist $\sum\limits_{n=2}^{\infty} f^n \in \boldsymbol{C}$. Infolgedessen ist sowohl $h = \sum\limits_{n=1}^{\infty} f^n \in \boldsymbol{L}$ als auch $\in \boldsymbol{L}^2$ da

$$\left|\sum_{n=1}^{\infty} f^n\right|^2 = \left|f + \sum_{n=2}^{\infty} f^n\right|^2 \leqq |f|^2 + \left|2f \sum_{n=2}^{\infty} f^n\right| + \left|\sum_{n=2}^{\infty} f^n\right|^2.$$

Damit ist dieser Satz bewiesen.

Nun betrachten wir auf dem Intervall $[0, \infty)$ definierte Funktionen, die nicht dort beschränkt sind, aber welche eine Singularität besitzen, so daß $|f(t)| \leqq M/t^\alpha$, $0 < \alpha < 1$ gilt.

Satz 3.3. *Gilt* $|f(t)| \leqq M/t^\alpha$, $0 < \alpha < 1$, *so folgt*

$$\sum_{n=1}^{\infty} f^n = \frac{f}{1-f}.$$

Beweis. Für $l f^n(t)$ gilt die folgende Abschätzung:

$$(3.3) \qquad |l f(t)| \leqq \frac{[M\Gamma(1-\alpha)\, t^{1-\alpha}]^n}{\Gamma(n(1-\alpha)+1)}, \qquad n = 1, 2, \ldots.$$

Offenbar ist diese Ungleichung gültig für $n = 1$. Der Beweis wird mittels Induktion geführt. Zunächst gilt

$$(3.4) \qquad |f^n(t)| \leqq \frac{[M\Gamma(1-\alpha)]^n\, t^{n(1-\alpha)-1}}{\Gamma(n(1-\alpha))}, \qquad n = 1, 2, \ldots$$

(wo $\Gamma(\alpha)$ die Gamma-Funktion bedeutet). Um (3.4) zu beweisen gilt

$$|f^{n+1}(t)| = |l f^n(t)| \leqq \int\limits_0 \frac{M(t-u)^{-\alpha}\, [M\Gamma(1-\alpha)]^n\, u^{n-n\alpha-1}}{\Gamma(u-u\alpha)}\, du$$

und setzen wir $u = tv$ in das Integral ein, so gilt

$$|f^{n+1}(t)| \leqq \frac{M^{n+1}\Gamma^n(1-\alpha)\,t^{n-n\alpha-\alpha}}{\Gamma(n-n\alpha)} \int\limits_0^1 (1-v)^{-\alpha}\,v^{n-n\alpha-1}\,dv$$

$$= \frac{M^{n+1}\Gamma^n(1-\alpha)\,t^{n-(n+1)\alpha}}{\Gamma(n-n\alpha)} \cdot \frac{\Gamma(1-\alpha)\,\Gamma(n-n\alpha)}{\Gamma(n-n\alpha+1-\alpha)}$$

$$= \frac{[M\,\Gamma(1-\alpha)]^{n+1}\,t^{(n+1)(1-\alpha)-1}}{\Gamma((n+1)(1-\alpha))},$$

woraus der Beweis von (3.4) und (3.3) sofort folgt. (Für ähnliche Abschätzungen siehe MIKHLIN [9] S. 18.) Wir nehmen irgendein festes $t_0 > 0$. Um zu beweisen, daß die Reihe $\sum\limits_{n=1}^{\infty} l\,f^n(t)$ gleichmäßig und absolut auf dem Intervall $0 \leqq t \leqq t_0$ konvergiert, genügt es zu zeigen, daß die Reihe, die a_n als allgemeines Glied hat, konvergent ist. Hier ist a_n das Glied auf der rechten Seite von (3.3), wobei t durch t_0 ersetzt ist. Hierzu braucht man nach CAUCHY's Kriterium[7] nur zu zeigen, daß $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{a_n} < 1$ ist.

In der Tat, mittels der Stirling-Formel gilt

$$\Gamma(\beta) = \frac{1}{\sqrt{2\pi\beta}}\,\beta^\beta \exp\left(-\beta + \frac{\vartheta}{12\beta}\right), \qquad 0 < \vartheta < 1.$$

Hieraus folgt

$$\sqrt[n]{a_n} = \frac{M\,t_0^{1-\alpha}\,\Gamma(1-\alpha)}{\Gamma^{1/n}(n(1-\alpha)+1)}$$

$$= \frac{M\,t_0^{1-\alpha}\,\Gamma(1-\alpha)\,[2\pi n(1-\alpha)+2\pi]^{1/2n} \exp\left(1-\alpha+\frac{1}{n}-\frac{\vartheta}{12n^2(1-\alpha)+12n}\right)}{[n(1-\alpha)+1]^{1-\alpha+1/n}}$$

und da dieser Ausdruck mit $1/n$ gegen Null strebt, ist der Satz bewiesen.

Diese drei Sätze bilden die theoretische Grundlage zu den folgenden Betrachtungen.

4. Integralgleichungen erster Art

Die Volterrasche Integralgleichung erster Art hat die Form

$$\int\limits_0^t k(t-u)\,f(u)\,du = g(t) \tag{4.1}$$

wo für $t \geqq 0$, k und g gegebene Funktionen sind und f gesucht ist. Hier hängt der Kern $k(t-u)$ nur von der Differenz $t-u$ ab. Es sei $k, g, f \in \boldsymbol{L}$. Die Operatorenmethode führt die Gl. (4.1) in eine gewöhnliche lineare algebraische Gleichung

$$k\,f = g \tag{4.2}$$

über, die im Operatorenkörper $\boldsymbol{K}$ die Lösung $f = g/k$ hat. Anschließend findet man die Lösung im Originalraum (falls diese existiert). Wenn die Integralgleichung (4.1) eine Lösung hat, so ist sie eindeutig bestimmt. Hätten wir nämlich zwei Lösungen f_1 und f_2, so wäre $k f_1 = g$ und $k f_2 = g$, woraus $k(f_1 - f_2) = \{0\}$ folgt. Der Satz von TITCHMARSH ergibt $f_1 = f_2$ oder $k = \{0\}$ was trivial wäre.

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[7] Zum Beispiel K. KNOPP, Theory and application of infinite series, S. 116. London 1946.

Ein *erstes Beispiel* für die Anwendung dieser Methode wird geliefert durch die Gleichung

$$\int_0^t \sin(t-u)\, f(u)\, du = g(t), \tag{4.3}$$

wo g gegeben ist, f unbekannt, $t \geqq 0$ ist. Nach (2.4) § 2 nimmt die Gl. (4.3) die Form

$$\frac{1}{s^2+1} f = g$$

im Operatorenkörper mit der Lösung $f = g + s^2 g$ an.

Wir haben also eine „verallgemeinerte Lösung" der Integralgleichung erhalten, d.h. eine Lösung, die ein Operator im Sinne von Mikusiński ist[8]. Diese Lösung läßt sich auf eine gewöhnliche Funktion reduzieren, wenn $s^2 g$ eine gewöhnliche Funktion ist. Ist die Ableitung g' total-stetig für $t \geqq 0$, so ergibt die Gl. (2.3), daß

$$s^2 g = g'' + g'(0) + s\, g(0)$$

ist. Ist nun auch $g(0) = g'(0) = 0$ so reduziert sich $s^2 g$ auf die Funktion g'' und wir haben als eindeutige Lösung von (4.3), $f(t) = g(t) + g''(t)$. In der Tat ist dies eine Lösung von (4.3) wie die Substitution von $f(t)$ in (4.3) zeigt.

Ein *zweites Beispiel* liefert die entsprechende Gleichung mit dem Kern

$$k(t) = \frac{t^{n-1}}{(n-1)!}, \qquad t \geqq 0. \tag{4.4}$$

Die Operatorenform dieser Gleichung ist $l^n f = g$ mit Lösung $f = s^n g$. Diese Lösung existiert immer als ein Operator falls g nur $\in \boldsymbol{L}$ ist. Ist aber auch die $(n-1)$-te Ableitung $g^{(n-1)}$ total-stetig und $g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0$, so hat die Integralgleichung (4.1) mit dem Kern (4.4) die eindeutige Lösung $f(t) = g^{(n)}(t)$ was eine gewöhnliche Funktion ist (s. Schmeidler [*15*], S. 210 für ein ähnliches Beispiel).

Die Integralgleichung

$$\int_0^t J_0(t-u)\, f(u)\, du = g(t), \qquad t > 0$$

wo $J_0(t)$ die Besselsche Funktion vom Index 0 ist, hat die eindeutige Lösung[9]

$$f(t) = \int_0^t \frac{J_1(t-u)}{t-u}\, g(u)\, du + g'(t)$$

falls g' total-stetig ist mit $g(0) = 0$. In der Tat, der Übergang zur Operatorenform ergibt

$$\{J_0(t)\}\{f(t)\} = g.$$

---

[8] Die klassischen Methoden liefern keine „verallgemeinerte Lösungen". Obwohl diese keine gewöhnlichen Funktionen sind, ist es jedoch öfters möglich, sie zu interpretieren, wie es z.B. bei der Diracschen Delta-Funktion möglich ist. Der Distributionenkalkül von Schwartz [*17*] liefert auch Lösungen, die Distributionen sind, welche sich in gewissen Fällen auf Funktionen reduzieren.

[9] Siehe auch: Schwartz [*18*] Livre VI, Transformations de Laplace, S. 21.

Da
$$\frac{1}{\{J_0(t)\}} = \sqrt{1+s^2} = \left(\sqrt{1+s^2} - s\right) + s = \left\{\frac{J_1(t)}{t}\right\} + s$$
([*11*] S. 163) ist, gilt
$$f = \left\{\frac{J_1(t)}{t}\right\} g + s g,$$

und der Beweis folgt wie in den obigen Beispielen.

Ist $k(t)$ und $g(t)$ n-mal differenzierbar ($n \geqq 1$) für $t \geqq 0$ und $k(0) = k'(0) = \cdots = k^{(n-2)}(0) = 0$ und $k^{(n-1)}(0) \neq 0$, so ergibt sich (s. auch [*3*] Bd. III, S. 151) durch n-malige Differentiation von (4.1) falls eine für $t > 0$ stetige Lösung $f(t)$ existiert,
$$\int_0^t k^{(n)}(t-u) f(u)\, du + k^{(n-1)}(0) f(t) = g^{(n)}(t)$$
für $t > 0$. Man erhält nach Division durch $k^{(n-1)}(0)$ hieraus eine Integralgleichung von *zweiter Art*, die nach der Methode von § 5 behandelt wird. Die obige Methode versagt z. B., wenn $k(t)$ in $t = 0$ keine Ableitungen besitzt, wie in dem Fall $k(t) = t^{-\alpha}$, $\alpha > 0$, was uns zu den singulären Integralgleichungen führt, die in § 8 betrachtet werden.

5. Integralgleichungen zweiter Art

Diese hat die Gestalt

$$\int_0^t k(t-u) f(u)\, du + g(t) = f(t), \qquad t > 0 \tag{5.1}$$

wo wieder f die gesuchte Funktion ist und $k, g \in \boldsymbol{L}$ bekannt sind. Diese Gleichung nimmt in der Operatorenform die Gestalt
$$k f + g = f$$
an, mit Lösung

$$f = g/(1-k). \tag{5.2}$$

Die Lösung der Gl. (5.1) ist wiederum eindeutig bestimmt falls sie existiert. Gäbe es zwei Lösungen, so würde ihre Differenz die Gleichung $k f_3 = f_3$ befriedigen, so daß $f_3 = \{0\}$ oder $k = 1$ wäre, was beidemal trivial ist. Da in der Gestalt (5.2) die Lösung meistens nicht unmittelbar[10] zum Originalraum zurückzuführen ist, schreiben wir die Lösung in der Form
$$f = g + \frac{k}{1-k} g$$
und entwickeln $k/(1-k)$ in eine Potenzreihe, indem wir einen der Sätze aus § 3 benutzen, je nachdem $k \in \boldsymbol{C}$ oder $k \in \boldsymbol{L}^2$ ist.

Satz 5.1. *Es sei* $f, g \in \boldsymbol{L}$ *und* $k \in \boldsymbol{C}$. *So hat die Integralgleichung* (5.1) *die Lösung*

$$f(t) = g(t) + \int_0^t h(u)\, g(t-u)\, du, \tag{5.3}$$

---

[10] Dagegen erhält man unmittelbar $f(t) = t$ als Lösung der Gl. (5.1) mit $k(t) = g(t) = \sin t$, $t > 0$.

wobei

$$h(t) = \sum_{n=1}^{\infty} k^n(t)$$

eine Reihe ist, die in jedem endlichen Intervall $0 \leqq t \leqq t_0$ *absolut und gleichmäßig konvergent ist und eine stetige Funktion darstellt.*

Der Beweis folgt sofort aus Satz 3.1 und es läßt sich zeigen, daß diese Lösung die Gl. (5.1) erfüllt.

Die Reihendarstellung von h ist in der allgemeinen Theorie der linearen Integralgleichungen als die „Neumannsche Reihe“ und die k^n sind als die „iterierten Kerne“ bekannt.

Auch gilt

Satz 5.2. *Wenn g und* $k \in \boldsymbol{L}^2$ *sind, so hat die Gl.* (5.1) *die Lösung* (5.3). *Hierbei ist* $h \in \boldsymbol{L}^2$ *mit* $f \in \boldsymbol{L}^2$.

Als Beispiel hierzu betrachten wir

$$\int_0^t e^{(t-u)} f(u)\,du + 2(t-1)\,e^{t^2} = f(t), \qquad t > 0.$$

In der Operatoren-Schreibweise gilt dann

$$\{e^t\}\{f(t)\} + 2\{(t-1)\,e^{t^2}\} = \{f(t)\}.$$

Hier ist

$$k = \{e^t\}, \qquad k^n = \left\{e^t \frac{t^{n-1}}{(n-1)!}\right\}, \quad \text{für} \quad n = 2, 3, \ldots$$

und

$$h = \left\{e^t\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots\right)\right\} = \{e^{2t}\}.$$

Nach Satz 5.1 folgt,

$$\begin{aligned} f(t) &= 2(t-1)\,e^{t^2} + \int_0^t e^{2(t-u)}\, 2(u-1)\, e^{u^2}\,du \\ &= 2(t-1)\,e^{t^2} + e^{2t}(e^{t^2-2t} - 1) \\ &= (2t-1)\,e^{t^2} - e^{2t} \end{aligned}$$

und in der Tat erfüllt diese Lösung die gegebene Gleichung.

Es ist zu bemerken, daß in diesem Beispiel die Anwendung der Laplace-Transformation nicht unmittelbar[11] möglich ist, da das Integral

$$\int_0^{\infty} e^{-st}(t-1)\,e^{t^2}\,dt$$

divergiert.

Beispiele die SCHMEIDLER ([*15*] S. 250) und TITCHMARSH ([*23*] S. 313) mittels der Laplace- bzw. Fourier-Transformationen löst sind auch vermöge der obigen Methode zu lösen.

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[11] Oder man müßte bei der Behandlung dieser Integralgleichung das „Fortsetzungsprinzip“ (s. [*3*] Bd. II, S. 259) benutzen.

6. Systeme von Integralgleichungen

Wir betrachten ein System Volterrascher Integralgleichungen zweiter Art

$$\begin{aligned} f_1(t) - \sum_{\nu=1}^{n} \int_0^t k_{1\nu}(t-u)\, f_\nu(u)\, du &= g_1(t) \\ \cdots\cdots\cdots&\cdots\cdots \\ f_n(t) - \sum_{\nu=1}^{n} \int_0^t k_{n\nu}(t-u)\, f_\nu(u)\, du &= g_n(t) \end{aligned} \tag{6.1}$$

wo die Funktionen $\{f_1(t)\}, \{f_2(t)\}, \ldots, \{f_n(t)\}$ unbekannt sind. Dieses System läßt sich mittels dieser Theorie in eleganter Weise auflösen, indem man die Matrizenrechnung benutzt. Setzen wir

$$\mathfrak{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \quad \mathfrak{G}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \quad \mathfrak{K}(t) = \begin{bmatrix} k_{11}(t) & k_{12}(t) & \ldots & k_{1n}(t) \\ k_{21}(t) & k_{22}(t) & \ldots & k_{2n}(t) \\ \vdots & \vdots & & \vdots \\ k_{n1}(t) & k_{n2}(t) & \ldots & k_{nn}(t) \end{bmatrix}$$

wo $\mathfrak{K}(t)$ die Kernmatrix des Systems (6.1) ist, so läßt sich das System in der Gestalt

$$\mathfrak{F}(t) - \int_0^t \mathfrak{K}(t-u)\, \mathfrak{F}(u)\, du = \mathfrak{G}(t) \tag{6.2}$$

schreiben. Die Matrizen sind Funktionen von t und das Integral einer Matrix ist diejenige Matrix, deren Elemente die Integrale der entsprechenden Elemente der gegebenen Matrix sind. Auch ist $\mathfrak{K}(t-u)\,\mathfrak{F}(u)$ im Sinne des Matrizenproduktes von $\mathfrak{K}(t-u)$ mit $\mathfrak{F}(u)$ zu verstehen. (Auf die Nichtkommutativität dieses Produktes ist zu achten.)

Nun löst man die Gl. (6.2) in ähnlicher Weise wie eine einzige Volterra-Integralgleichung zweiter Art auf. Ist $\mathfrak{F} = \{\mathfrak{F}(t)\}$ die Spaltenmatrix, deren Elemente nun $f_i = \{f_i(t)\}$, $i = 1, 2, \ldots, n$ sind, und gilt entsprechend für $\mathfrak{G} = \{\mathfrak{G}(t)\}$ und $\mathfrak{K} = \{\mathfrak{K}(t)\}$, so nimmt das System in der Operatorenform die Gestalt

$$\mathfrak{F} - \mathfrak{K}\mathfrak{F} = \mathfrak{G} \tag{6.3}$$

an, wobei

$$\left\{\int_0^t \mathfrak{K}(t-u)\, \mathfrak{F}(u)\, du\right\} = \{\mathfrak{K}(t)\}\{\mathfrak{F}(t)\} = \mathfrak{K}\mathfrak{F}$$

ist. Besteht die quadratische Matrix $\mathfrak{J}$ bis auf die Elemente 1 (der Einheitsoperator in $\boldsymbol{K}$) in der Hauptdiagonale, aus lauter Nullen $\{0\}$, so ist die Lösung von (6.3) formal durch

$$\mathfrak{F} = (\mathfrak{J} - \mathfrak{K})^{-1}\, \mathfrak{G}$$

gegeben, falls die Matrix $(\mathfrak{J} - \mathfrak{K})$ nicht-singulär ist (d.h. die Determinante $|\mathfrak{J} - \mathfrak{K}| \neq \{0\}$ ist), und $(\mathfrak{J} - \mathfrak{K})^{-1}$ die Kehrmatrix bedeutet, die durch die gewöhnlichen Methoden (z.B. KOWALEWSKI [8] S. 130) erhalten werden kann. Anschließend findet man die Lösung von (6.2) im Originalraum, falls sie existiert.

7. Integro-Differentialgleichungen und nicht-lineare Gleichungen

Technische[12] Aufgaben führen zuweilen auf Gleichungen, in denen die gesuchte Funktion außer unter einem Integral auch noch in ihren Differentialquotienten vorkommt. Die allgemeine Gleichung [*4*] dieser Art (vom Volterraschen Typ)

$$\alpha_n \frac{d^n f(t)}{d t^n} + \alpha_{n-1} \frac{d^{n-1}}{d t^{n-1}} f(t) + \cdots + \alpha_0 f(t) + \int_0^t k(t-u)\, f(u)\, du = g(t)$$

läßt sich mit den vorstehenden Mitteln lösen. Anstatt eine allgemeine Theorie zu entwickeln, illustrieren wir die Methode durch ein Beispiel, in dem wir folgende Gleichungen lösen:

$$f'(t) + f(t) + \int_0^t e^{(t-u)} f(u)\, du = \alpha \qquad (t > 0).$$

Ist $\{f(t)\}$ total-stetig und $f(0) = 0$, so gilt

$$s f + f + \frac{1}{s-1} f = \alpha l$$

oder $f = \alpha (s-1)/s^3$ woraus $f(t) = \alpha \left(t - \frac{t^2}{2}\right)$ folgt, welches tatsächlich die Gleichung erfüllt.

Die Operatorenmethode ist nicht nur auf *lineare* Integralgleichungen beschränkt, sondern auch anwendbar auf Gleichungen, in denen die unbekannte Funktion f mit sich selbst und mit gegebenen Funktionen mehrfach gefaltet vorkommt. Wir betrachten folgende quadratische Integralgleichungen (s. auch Poli [*14*]):

$$2 f(t) - \int_0^t f(t-u)\, f(u)\, du = \sin t.$$

Die Operatorenform dieser Gleichung

$$f^2 - 2 f + \frac{1}{s^2+1} = 0,$$

eine algebraische Gleichung zweiten Grades, hat die zwei Lösungen[12a] ([*11*] S. 159)

$$f = 1 \pm \sqrt{\frac{s^2}{s^2+1}} = 1 \pm s\{J_0(t)\}.$$

Da $s\{J_0(t)\} = \{J_0'(t)\} + J_0(0)$ und $J_0'(t) = -J_1(t)$, $J_0(0) = 1$ ist, sind die Lösungen

$$f = 1 \pm \left(-\{J_1(t)\} + 1\right).$$

Die eine Lösung $f = \{J_1(t)\}$ ergibt die gewöhnliche Funktion $f(t) = J_1(t)$ während die andere, nämlich $f = 2 - \{J_1(t)\}$, nicht auf eine gewöhnliche Funktion[13] reduzierbar ist.

Weitere Beispiele, die auch mittels dieser Methode lösbar sind, hat L. Poli [*14*] durch Laplace-Transformationen gelöst.

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[12] Zum Beispiel Leon Lichtenstein: Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integrodifferentialgleichungen. Berlin 1931.

[12a] Nicht jede algebraische Gleichung n-ten Grades ist im Operatorenkörper $\boldsymbol{K}$ lösbar (z.B. Beispiel in [*11*], S. 159).

[13] Hier entspricht der Operator 2, der Diracschen Delta-Funktion (mit der Zahl 2 multipliziert).

8. Singuläre Integralgleichungen

Hat der Kern einer Integralgleichung die Form $k(t) = j(t)/t^\alpha$, $\alpha > 0$, wo z.B. $j(t)$ stetig ist, so wird die entsprechende Gleichung *singulär* genannt. Nehmen wir zuerst den Fall $0 < \alpha < 1$, so hat das entsprechende Integral trotz des Unendlichwerdens des Integranden einen Sinn. Zuerst betrachten wir als Beispiel die *Abelsche* singuläre *Gleichung erster Art*[14]

$$\int_0^t \frac{f(u)}{(t-u)^\alpha}\,du = g(t) \qquad (0 < \alpha < 1)$$

für g eine total-stetige Funktion für $t \geqq 0$. Als Lösung ist

$$f(t) = \frac{\sin\alpha\pi}{\pi}\left[\int_0^t (t-u)^{\alpha-1} g'(u)\,du + g(0)\,t^{\alpha-1}\right].$$

In der Tat, hat die algebraische Gleichung

$$\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} f = g \qquad (0 < \alpha < 1)$$

die Lösung

$$\begin{aligned} f &= \frac{1}{\Gamma(1-\alpha)} l^\alpha s\, g \\ &= \frac{1}{\Gamma(1-\alpha)} l^\alpha\left[\{g'(t)\} + g(0)\right]. \end{aligned}$$

Da $[\Gamma(1-\alpha)\,\Gamma(\alpha)]^{-1} = (\sin\alpha\pi)/\pi$ ist, folgt die Behauptung.

Die *Abelsche Gleichung zweiter* Art

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{f(u)}{\sqrt{t-u}}\,du + \frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}} = f(t) \qquad (t > 0)$$

ist in der Operatorenform

$$l^{\frac{1}{2}} f + l^{\frac{1}{2}} = f$$

darstellbar. Da die Funktion $l^{\frac{1}{2}}$ die Voraussetzungen von Satz 3.3 erfüllt, nimmt die Lösung die Form

$$\begin{aligned} f &= l^{\frac{1}{2}} + \frac{l^{\frac{1}{2}}}{1 - l^{\frac{1}{2}}}\, l^{\frac{1}{2}} \\ &= l^{\frac{1}{2}} + \sum_{n=1}^{\infty} l^{\frac{n}{2}+\frac{1}{2}} \end{aligned}$$

an. Da weiterhin $l^\lambda = \left\{\frac{t^{\lambda-1}}{\Gamma(\tau)}\right\}$ für $\operatorname{Re}\lambda > 0$ ist ([*11*] S. 101), gilt

$$f(t) = \frac{1}{\sqrt{\pi t}} + \frac{1}{\sqrt{t}} \sum_{n=1}^{\infty} \frac{t^{2/n}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

---

[14] Wir nehmen in diesem Paragraphen an, daß f in jedem endlichen Intervall $0 < t_1 \leqq t \leqq t_0$, beschränkt ist. Dann existiert das Faltungsintegral für alle t und ist stetig für $t > 0$.

Nun gilt bekanntlich (z.B. [*26*] S. 507)

$$\sum_{n=1}^{\infty} \frac{t^{n/2}}{\Gamma\left(\frac{n+1}{2}\right)} = \sqrt{t}\, e^t \left(1 + \operatorname{erf} \sqrt{t}\right)$$

wo $\operatorname{erf} t = (2/\sqrt{\pi}) \int\limits_0^t e^{-u^2}\, du$, das Fehlerintegral bedeutet. Die Lösung der Integralgleichung nimmt also die Gestalt

$$f(t) = \frac{1}{\sqrt{\pi t}} + e^t \left(1 + \operatorname{erf} \sqrt{t}\right)$$

an. Diese Lösung erfüllt die gegebene Gleichung.

Bisher war $0 < \alpha < 1$. Hat aber der Kern die Form $j(t)/t^\alpha$, mit $\alpha = 1$, so müßte man das Integral der Gleichung, damit es überhaupt einen Sinn hat ($1/t$ ist nicht integrierbar), im Sinne des Cauchyschen Hauptwertes z.B. verstehen. Für den Fall $\alpha > 1$, könnte man z.B. das Integral im Sinne des *Hadamardschen* ,,partie finie" interpretieren (z.B. SCHWARZ [*17*] Bd. I, S. 38). Der Verf. hofft in einer späteren Arbeit singuläre Integralgleichungen dieser Art in Betracht zu ziehen.

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[15] Eine Übersetzung ins Englische ist vor kurzem erschienen. Die Vorlesungen von A. ERDÉLYI [*5*] geben eine elegante Darstellung der Theorie von MIKUSIŃSKI.

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(Eingegangen am 9. Juni 1958)

Asymptotic Theory of Second Order Differential Equations with Two Simple Turning Points

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Communicated by L. CESARI

1. Introduction[1]. In 1931 Langer initiated and gave the first of numerous contributions to what has become a successful theory for asymptotic expansions of the solutions of a differential equation with a turning point. This theory has been extended and applied to a great many questions by him and by others. An extensive list of references may be found in L. CESARI's book, *Asymptotic behavior and stability of solutions of differential equations*, Chapter IV, Springer-Verlag, Berlin, 1958.

In this paper we present a general asymptotic theory of ordinary second order linear differential equations with two simple turning points and containing a numerically large parameter. In particular, we are concerned with the asymptotic expansions with respect to complex λ of solutions of differential equations of the form

$$\frac{d^2 y}{d s^2} - \lambda^2 P(s, \lambda)\, y = 0. \tag{1.1}$$

We consider this equation for s in a closed, simply connected, perhaps unbounded region $\mathscr{D}_s$ of the complex plane. We assume that *for* $|\lambda| > N$ *and for* $s \in \mathscr{D}_s$ *the coefficient* $P(s, \lambda)$ *is of the form*[2]

$$P(s, \lambda) = \sum_0^\infty p_j(s)\, \lambda^{-j}, \tag{1.2}$$

where each p_j *is analytic*, and, most importantly, that $p_0(s)$ *has precisely two simple zeros*, α *and* β, *in the interior of* $\mathscr{D}_s$.

In a region which includes a turning point, *i.e.*, a zero or singularity of $p_0(s)$, the solutions of the differential equation (1.1) depend upon λ in so intricate a way as to have quite distinct asymptotic forms in different parts of the region, being dominant (exponentially large) in some parts and subdominant (exponentially small) in others. The asymptotic series for solutions of (1.1) over a region which includes just one simple turning point are known and are based upon Airy functions [*2*, *5*]. Thus, it would appear that the behavior of solutions of (1.1) over the entire region $\mathscr{D}_s$, which contains two simple turning points, is obtainable by the familiar procedure of evaluating the coefficients in the dependence relations which connect solutions whose behavior is known about one of the turning points with those solutions whose behavior is known about the other turning point. Unfortunately, this evaluation is not possible in general; two linearly independent solutions may have the same dominant asymptotic form, which makes the inference of the identity of two solutions from the identity of their asymptotic forms invalid. Thus, a new theory is necessary if we desire to have uniform asymptotic expansions of solutions of (1.1) over all $\mathscr{D}_s$. LANGER [*4*] has derived the leading terms of such asymptotic expansions in the special case where s is a real variable on a bounded interval, $p_0(s)$ is real-valued, and $p_1(s) \equiv 0$. We derive the asymptotic expansions to $n+1$ terms, where n is an arbitrary non-negative integer, of the solutions of (1.1) under the general hypotheses set forth in the first paragraph above together with some others of a more technical nature to be set forth later.

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[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contracts No. AF 18(600)-1481 and No. AF 49(638)-192.

[2] The letters M and N are always to be used as generic symbols for positive constants.

Interest in the problem discussed here stems mainly from possible applications for the theory derived. In certain regions, the differential equations for the angular and radial spheroidal functions are of the type (1.1). This is also true of the Whittaker equation for certain configurations of its parameters. The spheroidal functions are important in problems concerning scattering by a prolate spheroid. The Whittaker functions, disguised as Coulomb wave functions, occur in quantum mechanics. Equations of type (1.1) are also of interest in other problems of wave motion and diffraction.

We have divided the discussion below into three parts. In Part I we transform the differential equation (1.1) into one more suitable for analysis. We call this canonical form the *given equation*. We then give an algorithm for the construction of a *related equation* whose coefficients resemble the coefficients of the given equation to an arbitrarily prescribed degree. In Part II we study the solutions of the related equation. These involve Weber functions of large complex order and argument. They have been studied by ERDÉLYI, KENNEDY & MCGREGOR [*3*]. We make considerable use of their results and have shown that their asymptotic representations hold uniformly in $\arg \nu$ over a finite range of $\arg \nu$, where ν is the order of the Weber functions involved. We also give an algorithm for recursively determining the terms in the asymptotic expansions of these Weber functions. In Part III we prove that solutions of the related equation are asymptotic expansions to $n+1$ terms of solutions of the given equation, n being any non-negative integer. The method of proof is, as usual, to transform the given equation into an integral equation of Volterra type, whose kernel involves solutions of the related equation, and to solve this integral equation by the Picard method of successive iteration. We also give approximations for derivatives of solutions of the given equation. Our main results are stated as Theorems 1 and 2 of §§ 12 and 13. An especially interesting feature is the occurence of a denumerable number of characteristic solutions of the given equation. These are bounded and oscillatory on certain curves joining the turning points and exponentially small on their extensions beyond the turning points.

Part I: Construction of the Related Equation

2. The given equation. The analysis to follow involves certain functions and a mapping connected with $p_0(s)$. To simplify their form we adopt a normalized form of the differential equation (1.1) as the basis of the sequel. This normalization was used by LANGER in [*4*, §§ 2, 3]. Its adoption here will be seen to entail certain assumptions on $p_0(s)$.

We consider the mapping from the given region $\mathcal{D}_s$ onto a region $\mathcal{D}_z$ of the z-plane defined implicitly by the equation

$$(2.1)\qquad \int_{+1}^{z} (t^2-1)^{\frac{1}{2}}\,dt - \int_{\alpha}^{s} p_0^{\frac{1}{2}}(t)\,dt = 0.$$

An immediate question is whether or not (2.1) defines a mapping at all, and if so, is it a schlicht mapping? We can show that (2.1) defines a mapping and that at each point of $\mathcal{D}_s$ it defines a locally one-to-one mapping. The function which is the left member of (2.1) is analytic in z except at ± 1 and in s except at α and β. Its partial derivatives do not vanish except on the lines $z=\pm 1$ and $s=\alpha$ or β. Thus by the implicit function theorem for analytic functions, there exist in neighborhoods of all points of $\mathcal{D}_s$, except possibly α and β, analytic solutions $z(s)$ of (2.1) with inverses $s(z)$. Further, $ds/dz=(z^2-1)^{\frac{1}{2}}\,p_0^{-\frac{1}{2}}(s)$; and hence, $s'(z)\neq 0$ except perhaps at ± 1.

We now examine what happens at the exceptional points α and β. A computation shows that we may write

$$(2.2)\qquad \int_{1}^{z} (t^2-1)^{\frac{1}{2}}\,dt = (z-1)^{\frac{3}{2}}\,\Phi_1(z) \quad\text{and}\quad \int_{\alpha}^{s} p_0^{\frac{1}{2}}(t)\,dt = (s-\alpha)^{\frac{3}{2}}\,P_1(s),$$

where Φ_1 and P_1 are analytic in neighborhoods of $+1$ and α, respectively, and neither $\Phi_1(1)$ nor $P_1(\alpha)$ is zero. Thus, we may write

$$(2.3)\qquad F(z,s) = (z-1)\,\Phi_1^{\frac{2}{3}}(z) - (s-\alpha)\,P_1^{\frac{2}{3}}(s) = 0,$$

where F is analytic in a neighborhood of $(1,\alpha)$ and neither $\partial F/\partial z$ nor $\partial F/\partial s$ vanishes at $(1,\alpha)$. Therefore, by the implicit function theorem, there exists a solution $z(s)$ of (2.3), and hence of (2.1), with inverse $s(z)$, which is analytic in a neighborhood of α and such that $z(\alpha)=1$ and $s'(1)\neq 0$.

In order to apply this same technique to (2.1) at $s=\beta$, we must assume a normalization of the coefficient p_0. The integrals $\int_{1}^{-1} (t^2-1)^{\frac{1}{2}}\,dt$ and $\int_{\alpha}^{\beta} p_0^{\frac{1}{2}}(t)\,dt$ are independent of path provided the paths do not encircle the singularities of the integrands. However, it may be that the latter integral vanishes. We assume the contrary and choose the parameter so that[3]

$$(\mathrm{i})\qquad \int_{\alpha}^{\beta} p_0^{\frac{1}{2}}(t)\,dt = \int_{1}^{-1} (t^2-1)^{\frac{1}{2}}\,dt = -\tfrac{1}{2}\pi i.$$

We can now apply considerations analogous to those above to reach the conclusions that (2.1) has an analytic solution $z(s)$ at $(-1,\beta)$ with inverse $s(z)$

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[3] We do not consider the limiting behavior of the solutions of (1.1) as the turning points α and β approach each other. F. W. J. OLVER has informed the author by a private communication that he is examining this limiting behavior.

whose derivative s' is nonzero at -1. If $\arg p_0$ is constant on the line segment joining α and β, the hypothesis (i) is clearly fulfilled. This is the case treated by LANGER [4]; and since, in this instance, the mapping defined by (2.1) is one to one on the line segment joining α to β, it is schlicht in an open neighborhood of this closed segment.

We have shown that the implicit relation (2.1) defines a mapping from $\mathcal{D}_s$ onto a region $\mathcal{D}_z$ and that it defines a locally one to one mapping of some neighborhood of any point of $\mathcal{D}_s$ onto a neighborhood of $\mathcal{D}_z$. We require more than this and assume that

(ii) *The relation* (2.1) *defines a schlicht mapping*

$$z(s):\ \mathcal{D}_s \leftrightarrow \mathcal{D}_z,$$

with $[-1, 1] \subset \mathcal{D}_z$.

Under this assumption, the change of variables

$$s = s(z), \tag{2.4a}$$

$$y(s) = \left\{\frac{ds}{dz}\right\}^{\frac{1}{2}} u(z), \tag{2.4b}$$

where s satisfies (2.1), transforms the differential equation (1.1) into

$$\frac{d^2 u}{dz^2} - \lambda^2 Q(z, \lambda)\, u = 0, \tag{2.5}$$

with

$$Q(z, \lambda) = P\big(s(z), \lambda\big)\, s'^2 + \frac{1}{\lambda^2}\left\{\frac{3s''^2}{4s'^2} - \frac{s'''}{2s'}\right\} \qquad \left('=\frac{d}{dz}\right),$$

where Q is analytic for $|\lambda| > N$ and for $z \in \mathcal{D}_z$. We write

$$Q(z, \lambda) = \sum_0^\infty q_j(z)\, \lambda^{-j}.$$

In particular,

$$q_0(z) = (z^2 - 1).$$

We henceforward refer to the differential equation (2.5) as "the given equation".

3. The first approximating equation. The analysis of the given equation is based upon the construction of an equation which resembles it up to terms of the form $\lambda^{-n-1} O(1)$ in the coefficient of u, where n is any non-negative integer and $O(1)$ denotes a function of z and λ which is bounded for $z \in \mathcal{D}_z$ and for $|\lambda| > N$. The algorithm for the construction of this related equation is similar to the algorithm of MCKELVEY given in [6]. The matter at issue in MCKELVEY's paper is the approximation of solutions of an equation with a single turning point of order two. Such a turning point may be thought of as the confluence of two simple turning points so that similarities in the analyses of the two cases are not unexpected.

The point of beginning in both instances is WEBER's equation, which may be written in the form

$$\frac{d^2 V}{dt^2} + \left[\frac{\nu}{2} - \frac{t^2}{4}\right] V = 0.$$

We make the transformation

$$t = (2\lambda)^{\frac{1}{2}} z, \qquad V(t) = v(z),$$

and WEBER's equation becomes

$$\frac{d^2 v}{dz^2} - \lambda(\lambda z^2 - \nu)\, v = 0.$$

Let a non-negative integer n be chosen once for all. We now choose ν so that

$$\nu = \lambda - c - \sum_{0}^{n} c_j \lambda^{-j-1}, \tag{3.1}$$

where the (arbitrary) constants c and c_j are yet to be determined. This brings WEBER's equation into the form

$$\frac{d^2 v}{dz^2} - \left[\lambda^2 (z^2 - 1) + \lambda c + \sum_{0}^{n} c_j \lambda^{-j}\right] v = 0 \quad \text{or} \quad v'' - R(z, \lambda)\, v = 0, \tag{3.2}$$

which is our first approximating equation.

4. The second approximating equation. We next construct an equation resembling the given equation in both the λ^2 and λ terms of the coefficient of u. Formally, this step is almost identical with the corresponding one in [*6*]. The idea is to make a change of dependent variable

$$Z = \mu_0 v + \mu_1 v'/\lambda \tag{4.1}$$

in the first approximating equation (3.2) and to determine the functions μ_0 and μ_1 in such a way as to make the new differential equation in Z more closely resemble the given equation. By differentiating Z twice, employing (3.2) at each step to replace v'' by Rv, and by constructing the eliminant for v and v' among the three relations connecting Z, Z', and Z'' with v and v', we find that the differential equation satisfied by functions Z of the form (4.1) is

$$\frac{d^2 Z}{dz^2} - \frac{H}{\mathrm{D}_0} \frac{dZ}{dz} - \frac{J}{\mathrm{D}_0} Z = 0, \tag{4.2}$$

wherein

$$\mathrm{D}_0(z, \lambda) = \begin{vmatrix} \mu_0 & \mu_0' + \mu_1 R/\lambda \\ \mu_1/\lambda & \mu_0 + \mu_1'/\lambda \end{vmatrix}, \tag{4.3 a}$$

$$H(z, \lambda) = \begin{vmatrix} \mu_0 & \mu_0'' + 2\mu_1' R \lambda^{-1} + \mu_1 R'/\lambda \\ \mu_1/\lambda & 2\mu_0' + \mu_1''/\lambda \end{vmatrix}, \tag{4.3 b}$$

and

$$J(z, \lambda) = \begin{vmatrix} \mu_0'' + \mu_0 R + 2\mu_1' R \lambda^{-1} + \mu_1 R/\lambda & \mu_0' + \mu_1 R/\lambda \\ 2\mu_0' + \mu_1'' \lambda^{-1} + \mu_1 R/\lambda & \mu_0 + \mu_1'/\lambda \end{vmatrix}. \tag{4.3 c}$$

In so far as λ is concerned, the functions D_0 and H are bounded for $|\lambda| > N$ with $\mathrm{D}_0(z, \infty) = \mu_0^2 - \mu_1^2 q_0$.

We shall determine μ_0 and μ_1 by the condition that the terms in the coefficient J/D_0 in (4.2) which become infinite with λ have sum $\lambda^2 q_0(z) + \lambda q_1(z)$. We recall that the choice of the constant c in (3.2) is in our hands. Now, a simple division shows that

$$\frac{J}{\mathrm{D}_0} = R + \mathrm{D}_0^{-1} \begin{vmatrix} \mu_0'' + 2\mu_1' R \lambda^{-1} + \mu_1 R'/\lambda & \mu_0' + \mu_1 R/\lambda \\ 2\mu_0' + \mu_1''/\lambda & \mu_0 + \mu_1'/\lambda \end{vmatrix}. \tag{4.4}$$

Recalling the definition of R from (3.2), we observe that the coefficient of λ^2 in J/D_0 is q_0. The coefficient of λ may be computed and is

$$c + \mathrm{D}_0^{-1}(z, \infty) \begin{vmatrix} 2\mu_1' q_0 + \mu_1 q_0' & \mu_1 q_0 \\ 2\mu_0' & \mu_0 \end{vmatrix},$$

or

$$\mathrm{D}_0^{-1}(z, \infty)\left[(c\,\mu_0^2 + 2\mu_0\mu_1' q_0 + \mu_0\mu_1 q_0') - (c\,\mu_1^2 q_0 + 2\mu_0'\mu_1 q_0)\right].$$

Therefore, a sufficient condition that this coefficient be equal to q_1 is that μ_0 and μ_1 satisfy the following system of differential equations:

$$2\mu_1' q_0 + \mu_1 q_0' = \mu_0 (q_1 - c),$$

$$2\mu_0' = \mu_1 (q_1 - c),$$

with boundary condition that

$$\mu_0^2 - \mu_1^2 q_0 = \mathrm{D}_0(z, \infty) = 1.$$

A possible solution of this system is

$$\mu_0 = \cosh\vartheta, \qquad \mu_1 = \frac{\sinh\vartheta}{\varphi}, \tag{4.5 a}$$

where

$$\varphi(z) = (z^2 - 1)^{\frac{1}{2}}, \quad \arg\varphi = 0 \quad \text{for} \quad z > 1, \tag{4.5 b}$$

and

$$\vartheta(z) = \int_1^z \frac{q_1(s) - c}{2\varphi(s)}\, ds. \tag{4.5 c}$$

We deduce from these formulas that μ_0 and μ_1 are analytic in $\mathcal{D}_z$ except at $z = -1$ provided μ_1 is defined so as to be continuous at $z = 1$. Without proper choice of the constant c, μ_0 and μ_1 will fail to be analytic at -1. In order to render them analytic there, we choose c so that

$$c = \frac{\displaystyle\int_{-1}^{+1} \frac{q_1(s)}{\varphi(s)}\, ds}{\displaystyle\int_{-1}^{+1} \frac{ds}{\varphi(s)}}. \tag{4.6}$$

Then $\vartheta(-1) = 0$, and ϑ may be written in the form

$$\vartheta(z) = \int_{-1}^z \frac{q_1(s) - c}{2\varphi(s)}\, ds.$$

Provided μ_1 is now defined so as to be continuous at -1, this fact and computation show that μ_0 and μ_1 are analytic throughout $\mathcal{D}_z$.

The choice of μ_0 and μ_1 determine D_0, and from (4.5) it follows that

$$\mu_0^2 - \mu_1^2 q_0 = 1. \tag{4.7}$$

The relations (4.7) and (4.3) and the definition (3.2) of R imply that

$$\mathrm{D}_0(z,\lambda) = 1 + \lambda^{-1}(\mu_0\mu_1' - \mu_0'\mu_1 - c\,\mu_1^2) - \mu_1^2 \sum_0^n c_j \lambda^{-j-2}. \tag{4.8}$$

Thus if $\mathscr{D}_z$ is bounded, it is clear from (4.8) that D_0 is bounded away from zero for $|\lambda| > N$ and for $z \in \mathscr{D}_z$. If $\mathscr{D}_z$ is unbounded, we assume

(iii) D_0 *is bounded away from zero for* $z \in \mathscr{D}_z$ *and for* $|\lambda| > N$.

We observe directly from (4.3) that $H = \mathrm{D}_0'$. Therefore, we may make the change of variable

$$\zeta = \mathrm{D}_0^{-\frac{1}{2}} Z \tag{4.9}$$

to remove the first derivative term in equation (4.2). It then takes the form

$$\frac{d^2\zeta}{dz^2} - [\lambda^2(z^2-1) + \lambda q_1(z) + T(z,\lambda)]\,\zeta = 0, \tag{4.10a}$$

where

$$\lambda^2(z^2-1) + \lambda q_1 + T = \frac{J}{\mathrm{D}_0} + \frac{3}{4}\left(\frac{\mathrm{D}_0'}{\mathrm{D}_0}\right)^2 - \frac{\mathrm{D}_0''}{2\mathrm{D}_0}. \tag{4.10b}$$

The function T is analytic in $\mathscr{D}_z$ as R, μ_0, and μ_1 are all analytic there. Further, T is analytic in λ for $|\lambda| > N$; hence, we may write

$$T(z,\lambda) = \sum_0^\infty t_j(z)\,\lambda^{-j}. \tag{4.11}$$

We call the differential equation (4.10a) "the second approximating equation" for the given equation (2.5). It is important to note the way the constants c_j, which were introduced in (3.1), enter into the functions t_j. The c_j appear in T wherever R and R' do. From (4.4) and (4.3a), we observe that aside from the leading term R in the right member of (4.10b), R and R' have coefficients of order λ^{-1} or smaller in this right member. It follows that the constant c_i occurs linearly in t_i, always with a coefficient of $+1$, and is absent from every t_j with $j < i$.

5. The related equation. We are now in a position to construct a differential equation whose solutions are known and whose coefficients are identical with those of the given equation up to terms of order λ^{-n-1}. Inasmuch as this construction is already well known [*5*, *6*], we present only the results.

With ζ standing for any solution of equation (4.10a), we set

$$w = \mathrm{D}_1^{-\frac{1}{2}}\left(A\,\zeta + \frac{B\,\zeta'}{\lambda^2}\right), \tag{5.1a}$$

where

$$\begin{aligned} A(z,\lambda) &= \sum_0^n a_j(z)\,\lambda^{-j}, \\ B(z,\lambda) &= \sum_0^n b_j(z)\,\lambda^{-j}, \end{aligned} \tag{5.1b}$$

and

$$\mathrm{D}_1(z,\lambda)=\begin{vmatrix} A & A'+\dfrac{BS}{\lambda^2} \\ \dfrac{B}{\lambda^2} & A+\dfrac{B'}{\lambda^2} \end{vmatrix}, \tag{5.1 c}$$

and choose the functions a_j and b_j so that w satisfies a differential equation of the form

$$\frac{d^2 w}{d z^2}-\left[\lambda^2 Q(z,\lambda)-\frac{\Omega(z,\lambda)}{\lambda^{n+1}}\right] w=0, \tag{5.2}$$

where $\Omega(z, \lambda)$ is bounded for $z \in \mathscr{D}_z$ and for $|\lambda|>N$. The proper choices of the a_j's and b_j's are:

$$a_0(z)=1, \quad b_0(z)=\frac{1}{\varphi(z)}\int_1^z \frac{q_2(s)-t_0(s)}{2\varphi(s)}\,ds,$$

$$a_1(z)=0, \quad b_1(z)=\frac{1}{\varphi(z)}\int_1^z \left[(q_3-t_1)-2q_1 b_0'-q_1' b_0\right]\frac{ds}{2\varphi(s)},$$

(5.3) and for $j=2,\ldots,n$,

$$a_j(z)=\frac{1}{2}\int_1^z\left[b_{j-2}''+\sum_0^{j-2}(q_{j-k}-t_{j-k-2})\,b_k\right]ds,$$

$$b_j(z)=\frac{1}{\varphi}\int_1^z\Big[\sum_0^j(q_{j-k+2}-t_{j-k})\,a_k-(2q_1 b_{j-1}'+q_1' b_{j-1})-$$
$$-\sum_0^{j-2}(2t_{j-k-2}b_k'+t_{j-k-2}b_k)-a_j''\Big]\frac{ds}{2\varphi}.$$

In order to render the a_j's and b_j's analytic in $\mathscr{D}_z$, the constants c_j occurring in their defining expressions must be properly chosen. This may be done recursively since c_i occurs linearly in t_i and is absent from each t_j with $j<i$. The correct determinations are

$$c_0=\gamma\int_{-1}^1 (q_2-t_0^*)\frac{ds}{2\varphi}, \quad c_1=\gamma\int_{-1}^1\left[q_3-t_1^*-2q_1 b_0'-q_1' b_0\right]\frac{ds}{2\varphi},$$

(5.4) and for $j \geqq 2$

$$c_j=\gamma\int_{-1}^1\Big[\sum_1^j(q_{j-k+2}-t_{j-k})\,a_k-(2q_1 b_{j-1}'+q_1' b_{j-1})+(q_{j+2}-t_j^*)-$$
$$-\sum_0^{j-2}(2t_{j-k-2}b_k'+t_{j-k-2}b_k)-a_j''\Big]\frac{ds}{2\varphi},$$

where

$$t_j^*=t_j-c_j \quad \text{and} \quad \gamma=\left(\int_{-1}^1\frac{ds}{2\varphi}\right)^{-1}.$$

We now observe that the division by $\mathrm{D}_1^{\frac{1}{2}}$ in (5.1 a) is legitimate if $\mathscr{D}_z$ is bounded, since $\mathrm{D}_1(z,\infty)=1$. If $\mathscr{D}_z$ is unbounded, we assume

(iv) $D_1(z,\lambda)$ *is bounded away from zero for* $z\in\mathcal{D}_z$ *and for* $|\lambda|>N$.

The function $\Omega(z,\lambda)$ which appears in equation (5.2) is a function whose specific form may be computed, although it is of no interest to us. It involves the several functions A, B, Q, R, q_0, and q_1: It is, however, important to note that $\Omega(z,\lambda)$ is analytic in λ for $|\lambda|>N$ and is analytic in z for $z\in\mathcal{D}_z$. We henceforward refer to the differential equation (5.2) as the related equation. In Part II we shall single out certain of its solutions and describe their behavior for $z\in\mathcal{D}_z$ and $|\lambda|>N$.

Part II: Solutions of the Related Equation

6. Introduction. The construction just given of the related equation enables one to know the behavior of its solutions only as well as one knows the behavior of the Weber functions which are solutions of the first approximating equation (3.2). These are Weber functions of large complex order and unrestricted argument, *e.g.*, $D_{\frac{1}{2}(\nu-1)}(\sqrt{2\lambda}\,z)$. Their behavior is not yet completely known. ERDÉLYI, KENNEDY & MCGREGOR have derived the results most useful to us [3]. They have established approximations to the various functions

$$y_m(x,\nu)=D_{(\sigma\nu-1)/2}(\sqrt{2\nu}\,e^{\frac{1}{2}m\pi i}x),\qquad \sigma=(-1)^m, \tag{6.1}$$

with an error involving ν^{-1}. We strengthen their results by showing that their asymptotic representations hold uniformly in $\arg\nu$ over sectors of the ν-plane. We also extend their representations to asymptotic expansions, the terms of which are determined recursively by a quadrature at each step.

We obtain the behavior of solutions of equation (3.2) by letting $x=\sqrt{\frac{\lambda}{\nu}}\,z$ in the expansions for the functions (6.1). We then determine the asymptotic expansions over $\mathcal{D}_z$ of the solutions of the related equation. These are explicit up to terms involving ν^{-1}. While our results are limited by the lack of precise information on the structure of Weber functions of large complex order, approximations to terms involving ν^{1} are usually adequate in applications. When they are not, numerical methods for the calculation of the terms of the asymptotic expansions given in § 8 may be employed to give more precise approximations.

7. Domains in the x-plane. The determination of the regions of validity for the approximations to solutions of the given equation, which are derived in Part III, depends upon the character of a function $\Phi(x)$. It is defined by analytic continuation from its positive values on $(1,\infty)$, which are given by

$$\Phi(x)=\int_1^x(t^2-1)^{\frac{1}{2}}\,dt=\int_1^x\varphi(t)\,dt. \tag{7.1}$$

It is shown in [3, pp. 469, 470] that Φ is a schlicht mapping of a Riemann surface $\mathfrak{X}$ over the x-plane with branch points at ±1 onto a Riemann surface $\mathfrak{Z}$ over the Φ-plane whose branch points are the images of the branch points in $\mathfrak{X}$.

Clearly,

$$\Phi(x)=\tfrac{1}{2}\{x(x^2-1)^{\frac{1}{2}}-\ln[x+(x^2-1)^{\frac{1}{2}}]\}; \tag{7.2}$$

and for large $|x|$,

$$\Phi(x)=\pm\tfrac{1}{2}x^2[1+O(x^{-1})]. \tag{7.3}$$

The Weber functions (6.1) are entire functions of their argument and their order. Hence, in discussing them we may assume that

(7.4) The x-plane is cut from -1 to $+1$ and $|\arg \nu| \leqq \pi + \varepsilon, \quad \varepsilon > 0$.

These assumptions enable us to avoid considering more than a finite number of sheets of the logarithmic Riemann surface $\mathfrak{Z}$.

Let $\mathfrak{X}_1$ denote the x-plane cut in accordance with (7.4) and the condition $|\arg(x-1)| \leqq \pi$. The corresponding portion $\mathfrak{Z}_1$ of $\mathfrak{Z}$ consists of two sheets joined along the ray $\arg \Phi = 0$ (see Fig. 1). We now define certain regions of $\mathfrak{X}$ on which the behavior of the Weber functions (6.1) is conveniently describable. Let the index m, which appears in (6.1), have range 0, ± 1, ± 2, ± 3, and let

$$(7.5) \qquad \xi(x,\nu) = \nu\,\Phi(x).$$

Let γ_m be the point at infinity on $\mathfrak{Z}$ in the direction $\arg \Phi = -m\pi - \arg \nu$, and let ε be any positive number small enough to make the following definition meaningful. We define $\mathfrak{G}_m$ to be the largest closed, simply connected region $\mathfrak{G}$ of $\mathfrak{X}$ such that

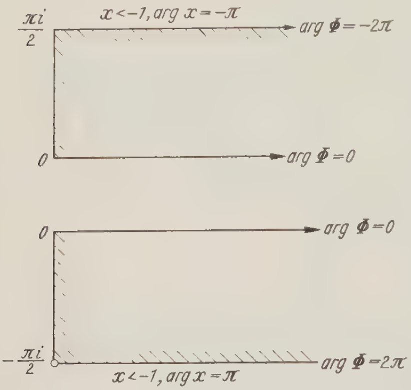

Fig. 1. The surface $\mathfrak{Z}_1$

(7.6a) $-\frac{3}{2}\pi + \varepsilon \leqq \arg \xi + m\pi \leqq \frac{3}{2}\pi - \varepsilon, \qquad x \in \mathfrak{G}$,

(7.6b) $|x+1| \geqq \varepsilon, \qquad x \in \mathfrak{G}$,

(7.6c) $|\Phi(x)| \geqq \varepsilon$, $x \in \mathfrak{G}$, except that $x=1$ and a neighborhood of $x=1$ may be in $\mathfrak{G}$.

(7.6d) Each point x in $\mathfrak{G}$ may be joined to the image of γ_m by an analytic, simple curve lying in $\mathfrak{G}$ and on which $\Re\xi$ is monotone (monotone increasing from the image of γ_m to x if m is odd, monotone decreasing if m is even).

The regions $\mathfrak{G}_m$ are illustrated in [3, p. 482]. In general, except for sectors with vertices at $x = \pm 1$ and widths 2ε in $\arg(x \pm 1)$ and the interior of the circle at $x = -1$ deleted by hypothesis (7.6b), each region $\mathfrak{G}_m$ covers the x-plane. Exception occurs, for example, when $m=0$ and $\arg \nu = 0$.

We denote by $\mathfrak{G}_m^*$ the reflection of $\mathfrak{G}_m$ in the origin; that is,

$$(7.7) \qquad \mathfrak{G}_m^* = \{x \mid e^{-\pi i} x \in \mathfrak{G}_m \quad \text{if} \quad \arg x \geqq 0, \quad e^{\pi i} x \in \mathfrak{G}_m \quad \text{if} \quad \arg x \leqq 0\}.$$

Now

$$(7.8) \qquad \Phi(e^{\pm \pi i} x) = \Phi(x) \mp \tfrac{1}{2}\pi i,$$

so that $\mathfrak{G}_{m\pm 2}^* \cup \mathfrak{G}_m$ is precisely $\mathfrak{G}_m$ plus all but a small sector with vertex at $x = -1$ and width 2ε in $\arg(x-1)$ of the neighborhood of $x=-1$ not in $\mathfrak{G}_m$. The definitions of $\mathfrak{G}_m$ and $\mathfrak{G}_m^*$ are essentially the same as those given in [3].

8. Asymptotic expansions of Weber functions. In this section we describe the asymptotic behavior of the Weber functions (6.1) over the regions $\mathfrak{S}_m$ and $\mathfrak{S}_m^*$. These functions are solutions of the transformed Weber equation

$$\frac{d^2 y}{d x^2} - \nu^2 (x^2 - 1)\, y = 0. \tag{8.1}$$

Since they are entire functions of x, $y_m = y_l$ if $m \equiv l \pmod 4$. The Wronskians of certain pairs of these solutions are [*1*, p. 42][4]

$$\mathscr{W}(y_m, y_{m\pm 2}, x) = \frac{2\sqrt{\pi}\,\nu\, e^{\frac{1}{2} m \pi i}}{\Gamma\left(\frac{-\sigma\nu + 1}{2}\right)}, \tag{8.2a}$$

$$\mathscr{W}(y_m, y_{m\pm 1}, x) = e^{\mp \frac{1}{4}\pi i (1+\sigma\nu)} \sqrt{2\nu}\, e^{\frac{1}{2} m\pi i}, \tag{8.2b}$$

where

$$\mathscr{W}(f, g, x) = f \frac{dg}{dx} - g \frac{df}{dx}, \tag{8.2c}$$

and

$$\sigma = (-1)^m. \tag{8.2d}$$

We henceforth adopt the convention that in formulas in which the double signs $\pm$ or $\mp$ appear, all upper signs or all lower signs are always to be used. It follows from (8.2) that any two of the four distinct solutions y_m are linearly independent except for $\nu = 0$ and certain other integral values of ν. If ν is not zero, and we always consider $|\nu|$ to be large, the solutions y_m and y_{m+1} are linearly independent without exception.

The asymptotic representations for the functions $y_m(x, \nu)$ involve the Airy function $\mathrm{Ai}(t)$, which is defined by the relation

$$\mathrm{Ai}(t) = \frac{\sqrt{t}}{\pi\sqrt{3}} K_{\frac{1}{3}}\left(\frac{2}{3} t^{\frac{3}{2}}\right), \tag{8.3}$$

in which $K_{\frac{1}{3}}$ is a modified Bessel function of the third kind [*3*, p. 463]. For each admissible m, the function

$$X_m(x, \nu) = \varphi^{-\frac{1}{2}}(x) \left(\frac{3}{4\sqrt{2}}\, \Phi(x)\right)^{\frac{1}{6}} \mathrm{Ai}\left[2\left(\frac{3\xi}{4\sqrt{2}}\right)^{\frac{2}{3}} \omega^m\right], \qquad \omega = e^{\frac{2}{3}\pi i}, \tag{8.4}$$

has a simple asymptotic expansion for $x \in \mathfrak{S}_m$. This is

$$X_m(x, \nu) = 2^{-\frac{5}{4}} \pi^{-\frac{1}{2}} \nu^{-\frac{1}{6}} \varphi^{-\frac{1}{2}}(x)\, e^{-\sigma\xi - m\pi i/6} \sum_0^N \frac{\Gamma(\nu + \frac{5}{6})\, \omega^{-3 r m/2}}{\Gamma(-\nu + \frac{5}{6})\, \nu!\, (2\xi)^r} + O(\xi^{-N-1}), \tag{8.5}$$

$$x \in \mathfrak{S}_m, \qquad m = 0, \pm 1, \pm 2, \pm 3.$$

The asymptotic series for dX_m/dx for $x \in \mathfrak{S}_m$ may be obtained from (8.5) by differentiation. ERDÉLYI, KENNEDY & MCGREGOR have shown that the functions (8.4) are asymptotic representations for the Weber functions (6.1) [*3*, p. 479]. These representations may be extended to asymptotic expansions by using the algorithm given by LANGER in [*5*] or similar algorithms. We use LANGER'S. In the notation of [*5*], it enables us to construct certain functions α_j and β_j

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[4] The formulas in [*1*] appear to contain misprints.

which are involved in the coefficients of the asymptotic expansions for the Weber functions (6.1). With reference to equation (8.1) the defining expressions of α_j and β_j specialize. They become:

$$\alpha_0(x)=1, \quad \beta_0(x)=\frac{1}{\varphi(x)}\int_1^x \frac{k(t)}{2\varphi(t)}\,dt, \tag{8.6a}$$
$$\alpha_1(x)=0, \quad \beta_1(x)=0,$$

and for $j\geqq 2$,

$$\alpha_j(x)=\frac{-1}{2}\int_1^x [\beta''_{j-2}(t)+k(t)\,\beta_{j-2}(t)]\,dt, \tag{8.6b}$$
$$\beta_j(x)=\frac{1}{\varphi(x)}\int_1^x \{\alpha''_j(t)+k(t)\,[\alpha_j(t)+2\beta_{j-2}(t)]-k'(t)\,\beta_{j-2}(t)\}\frac{dt}{2\varphi(t)},$$

wherein

$$k(x)=\frac{\Psi''(x)}{\Psi(x)} \quad \text{and} \quad \Psi(x)=\Phi^{\frac{1}{6}}(x)\,\varphi^{-\frac{1}{2}}(x). \tag{8.6c}$$

Since $\alpha_1=\beta_1=0$, $\alpha_{2j+1}=\beta_{2j+1}=0$ for all $j\geqq 0$. The following order relations are easily established. They are important in the extension of the asymptotic representations in [3] to asymptotic expansions. For $|x|>N$ and $j\geqq 0$,

$$\alpha_{2j}(x)=O(1), \quad \text{and} \quad \beta_{2j}(x)=O(x^{-1}). \tag{8.7}$$

The asymptotic expansions given below extend the results in [3]. We omit their derivation to avoid repetition of work in [3]. The α_j's and β_j's are sufficiently small at $x=\infty$ to guarantee uniform convergence of the integrals which enter into the derivation. This is the point in the extension of the analysis in [3] that requires careful attention. The bounds for the error terms in our expansions are uniform with respect to $\arg\nu$, if $\arg\nu$ lies on a bounded interval. This also applies to the results in [3]. The reason for this is that the limit a_m of integration in [3, equation (4.8)] may be fixed provided $\arg\nu$ has a range of length less than π. For example, if $|\arg\nu|\leqq\frac{1}{2}\pi-\varepsilon$, a_0 may be chosen as $x=e^{0i}\cdot\infty$; and if $|\arg\nu-\frac{1}{2}\pi|\leqq\frac{1}{2}\pi-\varepsilon$, a_0 may be chosen as $x=e^{-\frac{1}{4}\pi i}\cdot\infty$. For each such range of $\arg\nu$, a bound on the integral in [3, equation (4.8)] may be found which is independent of $\arg\nu$. It follows that for $\arg\nu$ bounded, bounds for the error terms independent of $\arg\nu$ can be found. This reasoning does not imply, however, that in the x-plane the regions of validity for the asymptotic expansions below are independent of $\arg\nu$. They are not, although in the neighborhood of infinity the boundaries of the regions of validity in the $\sqrt{\nu}\,x$-plane are asymptotically independent of $\arg\nu$.

$$y_m(x,\nu)=C_m[D_1^*(x,\nu)]^{-\frac{1}{2}}\times \tag{8.8a}$$
$$\times\left\{A^*(x,\nu)\,X_m(x,\nu)\,[1+O(\nu^{-i-1})]+\frac{B^*(x,\nu)}{\nu^2}\,\frac{dX_m(x,\nu)}{dx}\right\},$$

for $x\in\mathfrak{G}_m$; and

$$y_m(x,\nu)=C_m[D_1^*(x,\nu)]^{-\frac{1}{2}}\times \tag{8.8b}$$
$$\times\left\{A^*(x,\nu)\,X_m(x,\nu)+\frac{B^*(x,\nu)}{\nu^2}\,\frac{dX_m(x,\nu)}{dx}+O(\nu^{-i-1})\right\},$$

for all $x \in \{x \mid |x-1| < \varepsilon\}$ and $|\xi(x,\nu)| < N$. In these relations

$$C_m = 2\pi^{\frac{1}{2}} \nu^{\frac{1}{4}(\sigma\nu - \frac{1}{3})} e^{-\frac{1}{4}[\sigma\nu(1-m\pi i)+\frac{1}{3}m\pi i]}, \tag{8.9a}$$

$$\begin{aligned} &D_1^*(x,\nu) \\ &= \begin{vmatrix} A^*(x,\nu) & \nu^{2} B^*(x,\nu) \\ A^{*\prime}(x,\nu) - (x^2-1) B^*(x,\nu) + k(x) B^*(x,\nu)\nu^{-2} & A^*(x,\nu) + \nu^{-2} B^{*\prime}(x,\nu) \end{vmatrix}, \end{aligned} \tag{8.9b}$$

and

$$A^*(x,\nu) = \sum_0^{j-1} \alpha_i(x)\nu^{-i}, \qquad B^*(x,\nu) = \sum_0^{j-1} \beta_i(x)\nu^{-i}. \tag{8.9c}$$

The functions α_i, β_i, and k are defined by the relations (8.6), the functions $X_m(x,\nu)$ are described by the formulas (8.4) and (8.5), and $\sigma = (-1)^m$.

One uses the identity

$$y_m(x,\nu) = y_{m\pm 2}(e^{\mp\pi i} x, \nu) \tag{8.10}$$

to derive the form of $y_m(x,\nu)$ when $x \in \mathfrak{G}^*_{m\pm 2}$. Taken together the regions $\mathfrak{G}_m$ and $\mathfrak{G}^*_m$ $(m = 0, \pm 1, \pm 2, \pm 3)$ cover the x-plane, each point being in two regions having indices which differ by 1 or 3. Therefore, the expansions (8.8) yield the asymptotic behavior of two linearly independent solutions of the differential equation (8.1) for each x. We observe that zeros of the Airy function are not excluded from the region where the relation (8.8b) is fulfilled. This is the reason for the altered character of the error term in (8.8b) as compared with the error term in [*3*, equation (9.6)], for example. The behavior of dy_m/dx may be found via the relation

$$\frac{dD_\nu(x)}{dx} = \nu D_{\nu-1}(x) - \frac{1}{2} x D_\nu(x) \tag{8.11}$$

from the expansions (8.8).

It is interesting to note that if ν is real, all solutions of equation (8.1) are oscillatory on the interval $-1 < x < 1$. This occurs since, for the configuration of x and ν cited, $\Re(\xi(x,\nu)) = 0$.

9. Solutions of the related equation. The first approximating equation (3.2) has solutions

$$v_m(z,\lambda) = D_{\frac{1}{2}(\sigma\nu-1)}\left(\sqrt{2\lambda}\, e^{\frac{1}{2}m\pi i} z\right) \qquad (m = 0, \pm 1, \pm 2, \pm 3), \tag{9.1}$$

where $D_\nu(z)$ is the standard parabolic cylinder function of order ν [*1*], and where

$$\nu = \lambda - c - \sum_0^n c_j \lambda^{-j-1}. \tag{9.2}$$

Of course, $v_m = v_l$ if $l \equiv m \pmod 4$. The formula

$$v_m(z,\lambda) = y_m\left(\sqrt{\frac{\lambda}{\nu}}\, z, \nu\right) \tag{9.3}$$

relates the solutions of equations (3.2) and (8.1). In view of this connection, the asymptotic behavior of the solutions $v_m(z,\lambda)$ may now be easily described with reference to the x-plane. Since

$$x = \sqrt{\frac{\lambda}{\nu}}\, z, \tag{9.4}$$

the transition from regions on the x-plane to their images on the z-plane is elementary.

By directly applying the theory in [5] to equation (3.2), we could have described the behavior of certain solutions of (3.2), other than the solutions (9.1), on regions in the z-plane identical with the regions $\mathfrak{G}_m$ and $\mathfrak{G}_m^*$. However, the slight simplification in the description of the regions of validity of the asymptotic expansions, which is gained by this approach, is offset by an increase in difficulty in deriving these expansions, since equation (3.2) is superficially more complicated than equation (8.1). For this reason and to take fullest advantage of the work in [3], we proceeded as we have done.

We compute the Wronskians of the solutions $v_m(z, \lambda)$ using the formulas (8.2), (9.3), and (9.4). For future reference, the computed values are:

$$\mathscr{W}(v_m, v_{m\pm 2}, z) = \frac{2\sqrt{\pi\lambda}\, e^{\frac{1}{2}m\pi i}}{\Gamma\left(\frac{-\sigma\nu+1}{2}\right)},$$

$$\mathscr{W}(v_m, v_{m\pm 1}, z) = e^{\mp\frac{1}{4}\pi i(1+\sigma\nu)}\sqrt{2\lambda}\, e^{\frac{1}{2}m\pi i}. \tag{9.5}$$

To each solution $v_m(z, \lambda)$ of equation (3.2) there corresponds, through the relations (4.1), (4.9), and (5.1), a solution $w_m(z, \lambda)$ of the related equation. The solutions $w_m(z, \lambda)$ may be written in terms of the functions $v_m(z, \lambda)$ as follows:

$$w_m(z, \lambda) = E_0(z, \lambda)\, v_m(z, \lambda) + E_1(z, \lambda)\, v_m'(z, \lambda)/\lambda, \tag{9.6}$$

where

$$E_0(z, \lambda) = (D_0 D_1)^{-\frac{1}{2}}\left[A\,\mu_0 + \frac{B}{\lambda}\left(\frac{\mu_1 R}{\lambda^2} + \frac{\mu_0'}{\lambda} - \frac{\mu_0 D_0'}{2\lambda D_0}\right)\right],$$

$$E_1(z, \lambda) = (D_0 D_1)^{-\frac{1}{2}}\left[A\,\mu_1 + \frac{B}{\lambda}\left(\mu_0 + \frac{\mu_1'}{\lambda} - \frac{\mu_1 D_0'}{2\lambda D_0}\right)\right]. \tag{9.7}$$

Since

$$\mathscr{W}(w_l, w_m, z) = \mathscr{W}(v_l, v_m, z), \tag{9.8}$$

the linear independence of solutions w_l and w_m depends upon that of their correspondents v_l and v_m.

For the proofs to follow it is necessary to have at hand the asymptotic behavior, in first approximation, as $|\lambda| \to \infty$, of the solutions $w_m(z, \lambda)$ of the related equation together with the behavior of their first derivatives. One finds this behavior by considering the relations (9.6) in conjunction with the asymptotic expansions (8.8) and (8.5) and the relation (8.4). When $|\xi(x, \nu)| \leq M$ and $|x - 1| < \varepsilon$

$$w_m(z, \lambda) = C_m O(1), \quad \text{and} \quad w_m'(z, \lambda) = C_m O(\lambda^{\frac{2}{3}}). \tag{9.9a}$$

When $|\xi(-x, \nu)| \leqq M$ and $|x + 1| < \varepsilon$,

$$w_m(z, \lambda) = C_{m\pm 2} O(1), \quad \text{and} \quad w_m'(z, \lambda) = C_{m\pm 2} O(\lambda^{\frac{2}{3}}). \tag{9.9b}$$

If z is bounded, $x \in \mathfrak{G}_m \cup \mathfrak{G}_{m\pm 2}^*$, $|\xi(x, \nu)| > M$, and $|\xi(-x, \nu)| > M$, then

$$w_m(z, \lambda) = \Lambda_m(z, \lambda)\, O(\lambda^{-\frac{1}{4}}), \tag{9.10a}$$

$$w_m'(z, \lambda) = w_m(z, \lambda)\, O(\lambda), \tag{9.10b}$$

where

$$\Lambda_m(z, \lambda) = \nu^{\frac{1}{4}\sigma\nu}\, e^{-\frac{1}{4}\sigma\nu(1-m\pi i)}\, e^{-\sigma\xi}. \tag{9.10c}$$

If z is not bounded, then the functions E_0 and E_1 defined by (9.7) may be unbounded. In this case, the relations (9.10a) and (9.10b) are invalid. We therefore introduce a function $b(z)$ which is continuous, which is nonvanishing for $|z|>M$ and equal to 1 for $|z|\leq M$, and which is the "largest" function such that for each m

$$(9.11)\quad \left|b(z)\,\Lambda_m^{-1}(z,\lambda)\,w_m(z,\lambda)\right|<N\,|\nu|^{-\frac{1}{4}} \quad\text{and}\quad \left|b(z)\,\Lambda_m^{-1}(z,\lambda)\,\frac{w'_m(z,\lambda)}{\lambda}\right|<N\,|\nu|^{-\frac{1}{4}},$$

when $x\in\mathfrak{S}_m\cup\mathfrak{S}^*_{m\pm 2}$ and when $|\xi(x,\nu)|>M$ and $|\xi(-x,\nu)|>M$. Thus when the relations (9.11) apply,

$$(9.12\text{a})\qquad w_m(z,\lambda)=\frac{\Lambda_m(z,\lambda)\,O(\lambda^{-\frac{1}{4}})}{b(z)},$$

and

$$(9.12\text{b})\qquad w'_m(z,\lambda)=\frac{\Lambda_m(z,\lambda)\,O(\lambda^{\frac{3}{4}})}{b(z)}.$$

Part III: Asymptotic Expansions for Solutions of the Given Equation

10. Final hypotheses on $\mathcal{D}_z$. Our objective is to determine the asymptotic behavior throughout $\mathcal{D}_z$ of a pair of linearly independent solutions of the given equation (2.5). This will be done in the sections to follow. We first link the behavior of solutions of the given equation to that of solutions of the related equation by using the familiar method of variation of parameters. Its application to the given equation yields the integral equation

$$(10.1)\qquad u=w+\int_{z_*}^{z}\frac{w_a(z)\,w_b(t)-w_a(t)\,w_b(z)}{\mathscr{W}(w_a,w_b,t)}\,\frac{\Omega(t,\lambda)}{\lambda^{n+1}}\,u(t)\,dt,$$

which is equivalent to the given equation in the sense that an analytic solution of either one is a solution of the other. In equation (10.1), w_a and w_b may be any pair of linearly independent solutions of the related equation, w may be any solution of the related equation, and z_* may be any point in $\mathcal{D}_z$. The kernel in equation (10.1) is, of course, independent of the choice of w_a and w_b so that a unique solution u of the given equation is determined by specifying w and z_* in equation (10.1).

The variables z and x and the parameters λ and ν are always considered to fulfill the relations (9.4) and (9.2). Thus, given z and λ, ν and x are determined. In order to keep the investigation to a reasonable length and to avoid further notational complications, we make the assumption that

$$(\text{v})\qquad |\arg\lambda|\leqq\pi.$$

The second of the conditions (7.4) is thereby fulfilled when $|\lambda|$ is sufficiently large. There is no loss of generality in assuming that the first of these conditions is also satisfied. In what follows we fix λ, once for all, with $|\lambda|$ sufficiently large to fulfill all requirements placed upon it.

The following lemma will be used in the discussion of solutions of (10.1).

Lemma. *If*

a) *$\mathscr{A}$ is a closed, simply connected region (whose boundary may depend upon a complex parameter λ),*

b) *$f(z, \lambda)$ is continuous and bounded for $z \in \mathscr{A}$ and for $|\lambda| > N$,*

c) *$K(z, t, \lambda)$ is continuous for $z \in \mathscr{A}$, for $t \in \mathscr{A}$, and for $|\lambda| > N$,*

d) *there exist a constant M, a point $s \in \mathscr{A}$, and a set of rectifiable analytic curves joining the points of $\mathscr{A}$ to s such that for all $z \in \mathscr{A}$*

$$\int_s^z |K(z, t, \lambda)|\,|dt| < M,$$

then the solution of the integral equation

$$g(z, \lambda) = f(z, \lambda) + \lambda^{-k} \int_s^z K(z, t, \lambda)\, g(t, \lambda)\, dt \qquad (k > 0)$$

has the form

$$g(z, \lambda) = f(z, \lambda) + \lambda^{-k} O(1),$$

where $O(1)$ denotes a function of z and λ which is uniformly bounded for $z \in \mathscr{A}$ and for $|\lambda| > N$.

The hypotheses of the Lemma guarantee the uniform convergence of the series which is obtained by successive iteration of the integral equation, and from this the proof of the Lemma follows.

For the Lemma to apply to the integral equation (10.1), it is necessary that the kernel in (10.1) be bounded. We now make assumptions on $\mathscr{D}_z$ sufficient to establish the boundedness of the kernel. For each λ, let

$$\mathscr{R}_m = \{z \mid x(z) \in \mathfrak{S}_m \cup \mathfrak{S}^*_{m\pm 2}\} \qquad (m = 0,\ \pm 1,\ \pm 2,\ \pm 3). \tag{10.2}$$

In this definition and in the sequel, when a region with subscript larger than 3 or less than -3 appears, the region is understood to be the null set. Because of the overlapping of the regions $\mathfrak{S}_m$ and also of the regions $\mathfrak{S}^*_m$, one can show that $\mathscr{R}_m \subset \mathscr{R}_{m-1} \cup \mathscr{R}_{m+1}$ $(m = 0, \pm 1, \pm 2)$. It follows from the hypothesis (v) and the conditions (7.4) that we lose no generality by assuming that $\mathscr{R}_{-3} \subset \mathscr{R}_{-2}$ and $\mathscr{R}_3 \subset \mathscr{R}_2$. The regions $\mathscr{R}_m$ cover $\mathscr{D}_z$ and each point of $\mathscr{D}_z$ is in at least two such regions. Our final assumptions on $\mathscr{D}_z$ are:

(vi) *Corresponding to each region $\mathscr{R}_m$, there is a point z_m such that each point $z \in \mathscr{R}_m$ may be joined to z_m by an analytic curve Γ lying in $\mathscr{R}_m$ and such that on Γ, $\Re\,\xi$ is monotone (monotone increasing from z_m to z if m is odd, monotone decreasing from z_m to z if m is even).*

(vii) *The integrals*

$$\int_\Gamma \frac{\Omega(t, \lambda)\, dt}{b^2(t)}$$

are uniformly bounded for $z \in \mathscr{D}_z$ and for $|\lambda| > N$.

The function $\Omega(z, \lambda)$ first appears in equation (5.2). The function $b(t)$ is defined by the conditions (9.11). Of course, if $\mathscr{D}_z$ is bounded, hypothesis (vii) is automatically fulfilled.

There are certain shadow zones which must be excluded from the regions $\mathscr{R}_m$ in the discussion of subdominant solutions of the given equation. Their images on the x-plane adjoin the segment $-1 < x < 0$. Their size and presence depends upon arg λ. They are defined as follows:

$$\begin{aligned} \mathscr{S}_m = \{z \mid z \in \mathscr{R}_m,\ \Re(\sigma\xi) > N,\ |\xi(-x(z))| > N, \\ \text{and the paths joining } z \text{ to } z_m \text{ all pass through} \\ \text{the neighborhood of } \mathscr{R}_m \text{ where } |\xi(-x)| < N\}. \end{aligned} \tag{10.3}$$

The shadow zones are excluded for the reason that the subdominant solutions of the given equation undergo a change in their asymptotic forms as one crosses into them. The presence of these shadow zones (see Fig. 2 for an example) is a phenomenon which has no counterpart in the discussion of asymptotic solutions of differential equations having just one turning point.

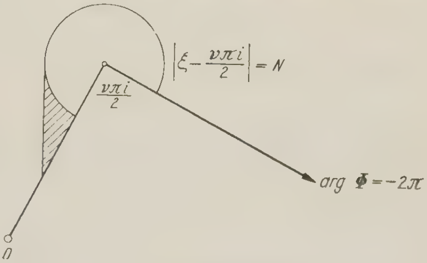

Fig. 2. A shadow zone

As we have remarked, we wish to determine the asymptotic structure throughout $\mathscr{D}_z$ of a pair of linearly independent solutions of the given equation. We first prove that, for each m, there exists a solution u_m which is asymptotically represented by w_m in the region $\mathscr{R}_m - \mathscr{S}_m$. We then derive connection formulas among the solutions u_m from which the behavior of the solutions u_m may be derived for all $z \in \mathscr{D}_z$. Lastly, we discuss certain characteristic solutions, which may well be of special interest in applications of the theory.

11. Subdominant solutions when $|\xi(\pm x, \nu)| > N$. For each region $\mathscr{R}_m - \mathscr{S}_m$ there is a solution of the given equation which is subdominant, exponentially small, for $z \in (\mathscr{R}_m - \mathscr{S}_m)$ and $\Re(\sigma\xi) > 0$. This solution is unique up to a constant factor. We now establish its existence and give its asymptotic properties. We assume for the moment that $z \in (\mathscr{R}_m - \mathscr{S}_m) \cap \mathscr{R}_{m+1}$. In equation (10.1) let $w_a - w_m$ and $w_b = w_{m+1}$, let $w = w_m$, and let the path of integration be a curve Γ which originates at z_m. For brevity, we adopt the notation $\tau = \xi(x(t), \nu)$ and $\xi = \xi(x(z), \nu)$. If the curve Γ should intersect one or both of the regions where $|\tau(\pm x)| \leqq N$, then $\Re(\sigma\xi) \leqq 0$. Thus, if we replace the subarcs of Γ on which $|\tau(\pm x)| \leqq N$ by arcs on which $|\tau(\pm x)| = N$, and if we note that on the remaining portion of Γ, $\Re(\sigma\xi) \leqq \Re(\sigma\tau)$, then we see that on the (new) path of integration $\exp[2\sigma(\xi - \tau)]$ is bounded.

The choices just made determine a solution u_m of equations (10.1) and (2.5). If we adopt the abbreviations

$$\begin{aligned} U_m(z) &= \Lambda_m^{-1}\, b(z)\, u_m(z, \lambda), \\ W_m(z) &= \Lambda_m^{-1}\, b(z)\, w_m(z, \lambda), \end{aligned} \tag{11.1}$$

where $\Lambda_m(z, \lambda)$ and $b(z)$ are defined by (9.10c) and (9.11), we can rewrite the integral equation (10.1) in the form

$$U_m(z) = W_m(z) + \int_\Gamma \{W_m(z)\, W_{m+1}(t) - W_m(t)\, W_{m+1}(z)\, e^{2\sigma(\xi - \tau)}\} \frac{O(1)\, \Omega(t)\, U_m(t)}{b^2(t)\, \lambda^{n+2}}\, dt. \tag{11.2}$$

In writing (11.2), we have used the evaluations (9.8) and (9.5) of $\mathscr{W}(w_m, w_{m+1}, z)$. The relations (9.10), (9.12) and (11.1) imply that for $|\xi(\pm x, \nu)| > N$ and

$z \in (\mathscr{R}_m - \mathscr{S}_m) \cap \mathscr{R}_{m+1}$, both W_m and W_{m+1} are bounded. We have already observed that $\exp[2\sigma(\xi - \tau)]$ is bounded on Γ. All functions in equation (11.2) are continuous. In virtue of these facts and hypothesis (vi) of § 10, the hypotheses of the Lemma are fulfilled with respect to equation (11.2). Thus for $z \in (\mathscr{R}_m - \mathscr{S}_m) \cap \mathscr{R}_{m+1}$, $|\xi(\pm x(z), \nu)| > N$, and $|\lambda| > N$,

$$U_m(z) = W_m(z) + \lambda^{-n-2} O(1). \tag{11.3}$$

We can show by analysis analogous to that above that this relation also holds for $z \in (\mathscr{R}_m - \mathscr{S}_m) \cap \mathscr{R}_{m-1}$. In terms of u_m and w_m, the result (11.3) takes the form

$$u_m(z, \lambda) = w_m(z, \lambda) + \frac{\Lambda_m(z, \lambda)\, O(1)}{b(z)\, \lambda^{n+2}} \qquad (|\lambda| > N). \tag{11.4}$$

Since $(\mathscr{R}_m - \mathscr{S}_m) \subset \mathscr{R}_{m+1} \cup \mathscr{R}_{m-1}$, (11.4) holds for all $z \in (\mathscr{R}_m - \mathscr{S}_m)$ such that $|\xi(\pm x(z), \nu)| > N$.

We can use the estimates (11.4) and (9.12) to obtain the the form of $u_m'(z, \lambda)$ directly from the equation found by differentiation of (10.1). It is

$$u_m'(z, \lambda) = w_m'(z, \lambda) + \frac{\Lambda_m(z, \lambda)\, O(1)}{b(z)\, \lambda^{n+1}}, \tag{11.5}$$

where the prime denotes differentiation with respect to z.

12. The solutions u_m when $|\xi(\pm x, \nu)| \leqq N$. There are two cases to be considered: $|\xi(x, \nu)| \leqq N$ and $|\xi(-x, \nu)| \leqq N$. For the moment we suppose that $|\xi| \leqq N$. We recall that the solution u_m of the given equation and equation (10.1) is that solution determined by the choices $w = w_m$ and $z_* = z_m$ in (10.1), the path of integration being a curve Γ. We divide Γ in two parts Γ_1 and Γ_2 on which $|\tau| \geqq N$ and $|\tau| < N$, respectively. As was done in § 11, we replace any part of Γ_1 on which $|\tau(-x)| < N$ by an arc on which $|\tau(-x)| = N$. We now write the integral equation for u_m in the form

$$u_m(z) = w_m(z) + \int_{\Gamma_1} K(z, t, \lambda)\, u_m(t)\, dt + \int_{\Gamma_2} K(z, t, \lambda)\, u_m(t)\, dt, \tag{12.1}$$

where

$$K(z, t, \lambda) = \frac{w_m(z)\, w_{m\pm1}(t) - w_m(t)\, w_{m\pm1}(z)}{\mathscr{W}(w_m, w_{m\pm1}, t)} \cdot \frac{\Omega(t, \lambda)}{\lambda^{n+1}},$$

the upper or lower signs being used according as $\Gamma_1 \subset \mathscr{R}_{m+1}$ or $\Gamma_1 \subset \mathscr{R}_{m-1}$.

For the sake of argument let $\Gamma_1 \subset \mathscr{R}_{m+1}$. On Γ_1, the behavior of u_m is then described by formula (11.4). Using this result and the relations (9.9a), (9.10), and (9.12), we find that

$$\begin{aligned} \int_{\Gamma_1} K(z, t, \lambda)\, u_m(t)\, dt &= C_m^2\, C_{m+1} \int_{\Gamma_1} \frac{[e^{\sigma\tau} O(1) - e^{-\sigma\tau} O(1)]}{e^{\frac{1}{4}\pi i \sigma \nu}} \cdot \frac{\Omega(t, \lambda)\, e^{-\sigma\tau} O(1)}{b^2(t)\, \lambda^{n+\frac{11}{6}}}\, dt, \\ &= \frac{C_m}{\lambda^{n+2}} \int_{\Gamma_1} [O(1) + e^{-2\sigma\tau} O(1)]\, \frac{\Omega(t, \lambda)}{b^2(t)}\, dt. \end{aligned}$$

But $\Re(\sigma\tau) \geqq 0$ on Γ_1, except perhaps for an arc on which $\Re(\sigma\tau) \geqq -N$. Therefore by hypothesis (vii),

$$\int_{\Gamma_1} K(z, t, \lambda)\, u_m(t, \lambda)\, dt = \frac{C_m O(1)}{\lambda^{n+2}}. \tag{12.2}$$

From the estimate (9.9a) and the order relation $\int\limits_{\Gamma_2} dt = O(\lambda^{\ \frac{2}{3}})$, we deduce that

$$\int\limits_{\Gamma_2} K(z, t, \lambda)\, dt = \lambda^{-(n+\frac{7}{3})} \int\limits_{\Gamma_2} \lambda^{\frac{2}{3}} O(1)\, dt = \lambda^{-(n+\frac{7}{3})} O(1). \tag{12.3}$$

The estimate (12.2) enables us to rewrite equation (12.1) as

$$\frac{u_m}{C_m} = \frac{w_m}{C_m} + \lambda^{-n-2} O(1) + \int\limits_{\Gamma_2} K(z, t, \lambda) \frac{u_m(t)}{C_m}\, dt.$$

The relation (12.3) shows that the hypotheses of the Lemma are fulfilled with respect to this equation; hence,

$$u_m(z, \lambda) = w_m(z, \lambda) + C_m \frac{O(1)}{\lambda^{n+2}} \qquad (|\xi(x, \nu)| \leqq N). \tag{12.4}$$

Since the order relation (9.9a) is independent of $\arg(z - \sqrt{\nu/\lambda})$, this result is valid in a full neighborhood of $x = 1$, or of $z = \sqrt{\nu/\lambda}$, where $|\xi| \leqq N$ and not just in that portion of $\mathscr{R}_m$ where $|\xi| \leqq N$.

We now consider the case $|\xi(-x)| \leqq N$. This situation is more complicated. If, for such z, $\Re(\sigma\xi) < -N$, then the estimate (12.2) no longer holds; and therefore we cannot determine the behavior of u_m when $|\xi(-x)| \leqq N$ as we did above for $|\xi| \leqq N$. We resolve this difficulty by showing that it is only an apparent one. Let us momentarily suppose that $m = \mp 2$ and $\Re(\sigma\xi) \leqq -N$. Then $\Re(\sigma\xi) \geqq N$ in the neighborhood of $\mathscr{R}_{m\pm 4}$ where $|\xi(-x)| \leqq N$. By the argument given in the next paragraph we are able to determine the behavior of $u_{m\pm 4}$ in a full neighborhood of $z = -\sqrt{\nu/\lambda}$, or of $x = -1$, where $|\xi(-x)| \leqq N$. We are also able to determine the behavior of $u_{m\pm 1}$ in such a neighborhood. Similar considerations apply to values of m other than ± 2. Thus, we shall always be able to determine the behavior of two linearly independent solutions of the given equation in a full neighborhood of $z = -\sqrt{\nu/\lambda}$ where $|\xi(-x)| \leqq N$.

As promised above, we now obtain the form of u_m if $|\xi(-x)| \leqq N$ and if, for such z, $\Re(\sigma\xi) \geqq -N$. To do this we replace $w_m(z)$ and $w_{m\pm 1}(z)$ in (12.1) by $w_{m\pm 2}(e^{\mp\pi i} z)$ and $w_{m\mp 1}(e^{\pm\pi i} z)$, respectively, and similarly replace $w_m(t)$ and $w_{m\pm 1}(t)$ for τ such that $|\tau(-x)| < N$. In virtue of the identity (8.10), the analysis used in the derivation of (12.4) may be easily modified to yield the conclusion that

$$u_m(z, \lambda) = w_{m\pm 2}(e^{\mp\pi i} z, \lambda) + \frac{C_{m\pm 2}\, O(1)}{\lambda^{n+2}}. \tag{12.5}$$

when $|\xi(-x, \nu)| \leqq N$ provided that for such z, $\Re(\sigma\xi) \geqq -N$. In (12.5) the upper signs are to be used if $m < 0$, lower signs if $m > 0$, and either if $m = 0$. As is the case when $|\xi| < N$, we need not impose the restriction here that z lie in $\mathscr{R}_m$. We need only suppose that the portion of the path of integration on which $|\xi(-x)| \leqq N$ is rectifiable. Thus the result (12.5) is valid in a full neighborhood of $z = -\sqrt{\nu/\lambda}$ where $|\xi(-x)| \leqq N$.

We can use the estimates of u_m obtained thus far to find the form of $u'_m(z, \lambda)$ when $|\xi| \leqq N$ or $|\xi(\ x)| \leqq N$ directly from the equation obtained by differentiation of (10.1). The results are:

$$u'_m(z, \lambda) = w'_m(z, \lambda) + \frac{C_m O(1)}{\lambda^{n+1}} \qquad (|\xi(x, \nu)| \leqq N), \tag{12.6a}$$

$$u'_m(z, \lambda) = w'_{m\pm 2}(e^{\mp\pi i} z, \lambda) + \frac{C_{m\pm 2}\, O(1)}{\lambda^{n+1}} \qquad (|\xi(-x, \nu)| \leqq N), \tag{12.6b}$$

in neighborhoods of $z = \sqrt{\nu/\lambda}$ and $z = -\sqrt{\nu/\lambda}$, respectively.

We summarize our results in a theorem:

Theorem 1. *Under the hypotheses* (i) *through* (vii) *and corresponding to each region* $\mathscr{R}_m$, *the given equation* (2.5) *has a solution* u_m *with the following asymptotic behavior when* $|\lambda| > N$:

a) *When* $z \in (\mathscr{R}_m - \mathscr{S}_m)$ *and* $|\xi(\pm x, \nu)| > N$, u_m *is exponentially large or small according as* $\Re(\sigma\xi) < 0$ *or* $\Re(\sigma\xi) > 0$ *and is described by the formulas* (11.4) *and* (11.5).

b) *When* $|z - \sqrt{\nu/\lambda}| \leqq \varepsilon$ *and* $|\xi(x, \nu)| \leqq N$ *or* $|z + \sqrt{\nu/\lambda}| \leqq \varepsilon$ *and* $|\xi(-x, \nu)| \leqq N$, u_m *is oscillatory in character and is described by the relations* (12.4), (12.6a) *and* (12.5), (12.6b), *respectively, provided* $\sqrt{\nu/\lambda}$ *and* $-\sqrt{\nu/\lambda}$ *lie in* $\mathscr{R}_m$ *and* $\Re(\sigma\xi) \geqq -N$.

c) *If* ν *is real, then* u_m *is bounded and oscillatory for* $-1 \leqq x \leqq 1$, $m = 0, \pm 1, \pm 2$.

The regions $\mathscr{R}_m$ and $\mathscr{S}_m$, the variables x and ξ, and the parameters ν and σ are defined by (10.2), (10.3), (9.4), (7.5), (9.2), and (6.1), respectively. The functions and constants appearing in the relations (11.4), (11.5), (12.4), (12.5) and (12.6) are defined in Parts I and II. The bounds for the error terms in all of the estimates for solutions of the given equation which have been derived are uniform in $\arg\lambda$ for $|\arg\lambda| \leqq \pi$. This is so for the same reasons as those cited in § 8 for the uniform boundedness of the error terms in the expansions (8.8a) and (8.8b) with respect to $\arg\nu$.

13. Connection formulas. It remains to give a description of the solutions $u_m(z, \lambda)$ when $|\xi(\pm x)| > N$ and when z is not confined to the specific region $\mathscr{R}_m$ as assumed in Theorem 1. The dependence relation among three solutions u_l, u_m, and u_{m+1} of the given equation may be written in the form

$$u_l = A_{l,m} u_m + B_{l,m} u_{m+1}. \tag{13.1}$$

The coefficients are, of course, given by the formulas

$$A_{l,m} = \frac{\mathscr{W}(u_l, u_{m+1}, z)}{\mathscr{W}(u_m, u_{m+1}, z)}, \qquad B_{l,m} = \frac{\mathscr{W}(u_m, u_l, z)}{\mathscr{W}(u_m, u_{m+1}, z)}. \tag{13.2}$$

Since the given equation lacks a first derivative term, the Wronskians in (13.2) are all constants and may be evaluated at $z = \sqrt{\nu/\lambda}$, where $x = 1$ and $\xi = 0$. If each of the solutions u_l, u_m, and u_{m+1} fulfills the relation (12.4), then we may conclude that

$$\mathscr{W}(u_k, u_j) = \mathscr{W}(w_k, w_j) + C_k C_j O(\lambda^{-n-1}),$$

where $k, j = l, m$, or $m+1$. Thus by relation (9.8),

$$\begin{aligned} A_{l,m} &= \frac{\mathscr{W}(v_l, v_{m+1})}{\mathscr{W}(v_m, v_{m+1})} + C_{m+1} C_l e^{\frac{1}{4}\sigma\pi i\nu} O(\lambda^{-n-3/2}), \\ B_{l,m} &= \frac{\mathscr{W}(v_m, v_l)}{\mathscr{W}(v_m, v_{m+1})} + C_m C_l e^{\frac{1}{4}\sigma\pi i\nu} O(\lambda^{-n-3/2}). \end{aligned} \tag{13.3}$$

Since $v_k = v_j$ if $k \equiv j \pmod 4$, the relations (9.5) always suffice to give the values of the Wronskians in (13.3). The constants C_j are defined by (8.9a).

The indices l and m for which the formulas (13.1) and (13.3) hold depend upon $\arg\nu$; hence, the pairs of linearly independent solutions of the given equation whose behavior can be determined over all of $\mathscr{D}_z$ depend upon $\arg\nu$. The selection of these fundamental pairs of solutions may be made using Theorem 1 and the connection formulas (13.1). We omit the details of the selection and state only the results.

Theorem 2. *For each value of λ such that $|\arg\lambda| \leqq \pi$, the given equation* (2.5) *has a pair of linearly independent solutions whose asymptotic behavior over all of $\mathcal{D}_z$ may be found from Theorem* 1 *and the connection formulas* (13.1); *namely,*

$$\text{if} \quad \left.\begin{array}{r} 0 \leqq \arg\nu < \pi \\ \pi \leqq \arg\nu \leqq \pi + \varepsilon \\ -\pi < \arg\nu \leqq 0 \\ -\pi - \varepsilon \leqq \arg\nu \leqq -\pi \end{array}\right\} \text{ the fundamental pair is } \left\{\begin{array}{l} u_{-2}, u_1 \\ u_{-3}, u_0 \\ u_{-1}, u_2 \\ u_0, u_3. \end{array}\right.$$

14. Characteristic solutions. If $\frac{1}{2}(\sigma\nu - 1) = k$, a non-negative integer,

$$y_m(x, \nu) = e^{-\frac{1}{2}(2k+1)x^2} H_k\left(\sqrt{4k+2}\,x\right),$$

where $y_m(x, \nu)$ is a solution (6.1) of the transformed Weber equation (8.1) and $H_k(x)$ is the Hermite polynomial of order k. The Hermite polynomials are even or odd according to the parity of k so that when $\frac{1}{2}(\sigma\nu - 1) = k$,

$$y_m(x, \nu) = (-1)^k\, y_{m\pm 2}(x, \nu).$$

These facts are reflected in the behavior of the solutions of the given equation. If $|\arg\lambda| < \varepsilon$,

$$\mathscr{W}(u_0, u_{\pm 2}) = \mathscr{W}(y_0, y_{\pm 2}) + \nu^{(\nu-\frac{1}{3})/2}\, e^{-\frac{1}{2}\nu}\, e^{\pm\nu\pi i}\left[O(\lambda^{-n-1}) - O(\lambda^{-n-1})\right].$$

where the upper or lower signs are to be used according as $\arg\nu < 0$ or $\arg\nu > 0$. When $\nu = 2k + 1$, $\mathscr{W}(y_0, y_{\pm 2}) = 0$. Thus, as shown by Langer [*4*], for certain values of λ the Wronskian $\mathscr{W}(u_0, u_{\pm 2})$ vanishes. These characteristic values are countable in number and are given by the formula

$$\lambda_k = 2k + 1 + O(\lambda_k^{-n-1}).$$

At the characteristic values,

$$u_0(z, \lambda_k) = (-1)^k\, u_{\pm 2}(z, \lambda_k)\left[1 + O(\lambda_k^{-n-1})\right],$$

where the upper or lower signs are to be used according as $\arg(\nu(\lambda_k))$ is negative or positive. We call $u_0(z, \lambda_k)$ a characteristic solution of the given equation. We can observe the unique behavior of a characteristic solution $u_0(z, \lambda_k)$ for real z by referring to Theorem 1.

Since $|\Re(\lambda_k \Phi)| < N$ for values of z such that $-1 \leqq \Re z \leqq 1$ and $|\Im z| < N|\lambda_k|^{-1}$, a characteristic solution is oscillatory for z in a strip of width $O(\lambda_k^{-1})$ centered on the interval $[-1, 1]$. In certain regions outside of such a strip, a characteristic solution is subdominant and decreases exponentially to zero as $|z| \to \infty$. These regions are sectors of width $\frac{1}{2}\pi - \varepsilon$ with vertices at $z = \pm\sqrt{\lambda_k/\nu(\lambda_k)}$ and which are centered on the real axis. In particular, a characteristic solution is uniformly bounded for real z. Only characteristic solutions and their multiples have this property.

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(Received June 10, 1958)

On Singularities of Solutions of Partial Differential Equations in Three Variables

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Communicated by S. BERGMAN

1. Introduction

In the case of linear partial differential equations in two independent variables there exist various operators which transform analytic functions of a complex variable into complex or real solutions of these differential equations. Such mappings of the algebra of analytic functions into the linear space of the aforementioned solutions can be considered as a natural generalization of the well-known relation between analytic functions and harmonic functions in two variables. Some of these operators preserve many properties of the analytic functions involved. BERGMAN operators [*1*]★ possess this important feature. In this way theorems on analytic functions can be "translated" into theorems on solutions of partial differential equations in two variables, *i.e.* the theory of these solutions can be based on the theory of analytic functions in a systematic manner.

In the case of elliptic partial differential equations in three variables with analytic coefficients there exist similar possibilities. For example, solutions of LAPLACE'S equation

$$\Delta v = v_{x_1 x_1} + v_{x_2 x_2} + v_{x_3 x_3} = 0$$

can be obtained from functions $f(u, \zeta)$ of two complex variables $\zeta = e^{it}$ and

$$u = x_1 + \tfrac{1}{2}\left((x_3 + i\,x_2)\,\zeta - \tfrac{1}{2}\,(x_3 - i\,x_2)\,\zeta^{-1}\right)$$

by means of the WHITTAKER representation [*2*]

$$v(x_1, x_2, x_3) = \frac{1}{2\pi}\int_0^{2\pi} f(u, e^{it})\,dt = \frac{1}{2\pi i}\int_{|\zeta|=1} f(u,\zeta)\,\zeta^{-1}\,d\zeta\,.$$

Using this representation, BERGMAN [*3—6*] investigated harmonic functions v corresponding to different classes of "*associated functions*" f, and solutions of equation (1.1), below.

The present paper concerns partial differential equations in three variables of the type

$$v_{x_1 x_1} + v_{x_2 x_2} + v_{x_3 x_3} + \alpha(x_2, x_3)\,v = 0. \tag{1.1}$$

This work was supported by the United States Air Force through the Office of Scientific Research and Development Command. Contract No. AF 49(638)-362, Proj. Nr. 812.

★ *Cf.* the bibliography at the end of this paper.

We mention that our investigation can be extended to certain classes of more general partial differential equations in three variables with analytic coefficients whose solutions exhibit a similar behavior. The basic tools to show this are Bergman operators.

Before stating in Section 3 the problem considered in this paper, we introduce suitable notation.

2. Representation of solutions of (1.1)

We assume that the coefficient $\alpha(x_2, x_3)$ in (1.1) is an analytic function. If we introduce new variables

$$X = x_1, \quad Z = \tfrac{1}{2}(x_3 + i x_2), \quad Z^* = -\tfrac{1}{2}(x_3 - i x_2)$$

and use

$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial Z^*} = -\frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_2},$$

equation (1.1) takes the form

$$w_{XX} - w_{ZZ^*} + F(Z, Z^*)\, w = 0 \tag{2.1}$$

where $w(X, Z, Z^*) \equiv v(x_1, x_2, x_3)$ and $F(Z, Z^*) \equiv \alpha(x_2, x_3)$.

We assume that $F(Z, Z^*)$ is an entire function.

Theorem 1 (BERGMAN [7]). *Let $f_n(Z)$ be an analytic function which is regular in a simply-connected domain $\mathfrak{D}$ containing the origin. Let the functions*

$$q_{nsp}(Z, Z^*), \quad s = n - 2\nu, \quad \nu = 0, 1, \ldots, [\tfrac{1}{2}n], \quad p = \nu, \nu + 1, \ldots, \quad n = 0, 1, \ldots,$$

$$q_{nn0} \equiv 0,$$

be solutions of the system

$$(1 - \delta_{sn})(s + 2)(s + 1)\, q_{n,s+2,p} - \frac{\partial^2 q_{nsp}}{\partial Z\, \partial Z^*} - \frac{\partial q_{n,s,p+1}}{\partial Z^*} + (q_{nsp} + \delta_{p0})\, F = 0 \tag{2.2}$$

with the property

$$q_{nsp}(Z, 0) = 0. \tag{2.3}$$

Then the functions

$$\begin{aligned} w_n(X, Z, Z^*; f_n) &= P_n(f_n) \\ &= \sum_{\nu=0}^{[\frac{1}{2}n]} X^{n-2\nu} \Big(\delta_{0\nu} f_n(Z) + \sum_{p=\nu}^{\infty} q_{n,n-2\nu,p}(Z, Z^*) \int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{p-1}} f_n(Z_p)\, dZ_p \ldots dZ_1\Big) \end{aligned} \tag{2.4}$$

are solutions of (2.1) *which are regular for $Z \in \mathfrak{D}$, $|X| < \infty$. δ_{sn} etc.* are Kronecker symbols.

Solutions which are real for real values of the original variables

$$x_1 = X, \quad x_2 = -i(Z + Z^*), \quad x_3 = Z - Z^*$$

will be termed *real solutions* and denoted by capital letters. Obviously,

$$W_n(X, Z, Z^*; f_n) = \tfrac{1}{2}\big(w_n(X, Z, Z^*; f_n(Z)) + \overline{w}_n(X, Z^*, Z; \overline{f}_n(Z^*))\big) \tag{2.5}$$

is a real solution corresponding to the complex solution $w_n(X, Z, Z^*; f_n)$; *cf.* (2.4).

Remark. We shall consider solutions of (2.1). In this connection it is important to study what properties of those solutions are completely independent of the special form of the coefficient F or depend only on certain properties of F. Our aim is to work out tools for investigating those properties. In some cases of differential equations (2.1) our problem can be reduced to a similar one concerning certain differential equations in *two* variables. Then we may derive the desired relations by means of previous results [*10*].

3. The problem considered in this paper

The representation (2.4) is helpful in considering coefficient problems, *i.e.*, in determining properties of solutions of (2.1) in terms of the coefficients of the corresponding power series development. Applications of this type have been discussed by Bergman [*7*].

The properties of the operator $P_n(f_n)$ are similar to those of the so-called *Bergman operator of the first kind* introduced by Bergman [*8*] for generating solutions of partial differential equations in two variables. Indeed, there exist simple relations between the solution $w_n(X, Z, Z^*; f_n)$ and its "*associated function*" $f_n(Z)$. From (2.3) and (2.4) we obtain

$$w_n(X, Z, 0; f_n) = X^n f_n(Z), \tag{3.1 a}$$

$$w_n(X, 0, Z^*; f_n) = X^n f_n(0). \tag{3.1 b}$$

In the case of the corresponding real solution (2.5) we thus have

$$W_n(X, Z, 0; f_n) = \tfrac{1}{2} X^n \big(f_n(Z) + \bar{f}_n(0)\big), \tag{3.2a}$$

$$W_n(X, 0, Z^*; f_n) = \tfrac{1}{2} X^n \big(f_n(0) + \bar{f}_n(Z^*)\big). \tag{3.2b}$$

We assume that $f_n(0)$ is real.

The relations (3.1) and (3.2) are basic in connection with coefficient problems. First of all, instead of $w_n(X, Z, Z^*)$ we may consider solutions of (2.1) of the form

$$w(X, Z, Z^*) = \sum_{n=0}^{N} w_n(X, Z, Z^*; f_n) \tag{3.3}$$

where the associated functions $f_0(Z), \ldots, f_N(Z)$ are regular in a common domain of the above type. Let

$$w(X, Z, Z^*) = \sum_{n=0}^{N} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} k_{nmp} X^n Z^m Z^{*p}. \tag{3.4}$$

Then from (3.1) it follows that

$$f_n(Z) = \sum_{m=0}^{\infty} k_{nm0} Z^m, \qquad n = 0, 1, \ldots, N. \tag{3.5}$$

Hence the domain of regularity, the nature and location of singularities and other properties of $w(X, Z, Z^*)$ can be determined by means of the $N+1$ subsequences (k_{nm0}), $m = 0, 1, \ldots$, $n = 0, 1, \ldots, N$, of the coefficients in the corresponding development (3.4). *Cf.* Bergman [*7*].

It may be expected that properties of $w(X, Z, Z^*)$ can be obtained in terms of other subsequences (k_{nmp}), $m = 0, 1, \ldots$, $n = 0, 1, \ldots, N$, $p > 0$ and fixed, of the coefficients of $w(X, Z, Z^*)$. The present paper is devoted to this problem.

4. Definition of the functions $a_p(X, Z)$

In order to investigate the problem just stated it suffices to establish relations between the $N+1$ subsequences (k_{nm0}), $m=0, 1, \ldots$, $n=0, 1, \ldots, N$, and the $n+1$ subsequences (k_{nmp}), $m=0, 1, \ldots$, $n=0, 1, \ldots, N$, where p is an arbitrary natural number. Relations of this type may then form the basis for a more general investigation of coefficient problems (*cf.* Sections 7 and 8). For this purpose we shall now introduce auxiliary functions $a_p(X, Z)$ in such a manner that relations between the above subsequences of coefficients are equivalent to relations between the functions $a_p(X, Z)$.

Lemma 2. *Let $w(X, Z, Z^*)$ be an arbitrary solution of (2.1) which can be represented in the form (3.4). Let the coefficient $F(Z, Z^*)$ be an entire function which has the representation*

$$F(Z, Z^*) = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} c_{mp} Z^m Z^{*p}, \tag{4.1}$$

and let

$$c_p(Z) = \sum_{m=0}^{\infty} c_{mp} Z^m, \qquad p = 0, 1, \ldots. \tag{4.2}$$

Then the functions

$$a_p(X, Z) = \sum_{n=0}^{N} \sum_{m=0}^{\infty} k_{nmp} X^n Z^m, \qquad p = 0, 1, \ldots, \tag{4.3}$$

are solutions of the following system of partial differential equations:

$$p \frac{\partial a_p}{\partial Z} - \frac{\partial^2 a_{p-1}}{\partial X^2} - \sum_{s=0}^{p-1} c_{p-s-1} a_s = 0, \qquad p = 1, 2, \ldots. \tag{4.4}$$

Proof. From (4.3) and (3.4) it follows that $w(X, Z, Z^*)$ can be represented in the form

$$w(X, Z, Z^*) = \sum_{p=0}^{\infty} a_p(X, Z) Z^{*p}. \tag{4.5}$$

From (4.1) and (4.2) we obtain

$$F(Z, Z^*) = \sum_{p=0}^{\infty} c_p(Z) Z^{*p}. \tag{4.6}$$

By substituting (4.5) and (4.6) into (2.1) we find

$$\sum_{p=0}^{\infty} \frac{\partial^2 a_p}{\partial X^2} Z^{*p} - \sum_{p=1}^{\infty} p \frac{\partial a_p}{\partial Z} Z^{*p-1} + \sum_{p=0}^{\infty} c_p Z^{*p} \sum_{s=0}^{\infty} a_s Z^{*s} = 0.$$

In order that this equation be satisfied for every value of Z^* the coefficient of each power of Z^* must vanish. This yields (4.4).

The coefficients in the development (4.3) of each function $a_p(X, Z)$ form just one of the above-mentioned sets of $N+1$ subsequences of the coefficients in the representation (3.4) of $w(X, Z, Z^*)$. Relations between the functions $a_p(X, Z)$ are thus equivalent to the desired relations between those sets of subsequences. These relations will be obtained from (4.4). The first r equations (4.4) involve only the functions $a_0(X, Z)$, $a_1(X, Z)$, $\ldots$, $a_r(X, Z)$.

5. Singularities of $a_r(X, Z)$, $r > 0$, and their relations to those of $a_0(X, Z)$

We consider any one of the functions (4.3), say $a_r(X, Z)$, $r>0$ and fixed, corresponding to a solution $w(X, Z, Z^*)$ of (2.1). Since we can determine the domain of regularity of $w(X, Z, Z^*)$ from the subsequences (k_{nm0}), $m=0, 1, \ldots$, $n=0, 1, \ldots, N$, (*cf.* (3.4)), *i.e.* from the function $a_0(X, Z)$, it is natural to inquire how the singularities of the above function $a_r(X, Z)$ are related to those of the function $a_0(X, Z)$.

First of all, the distribution of possible singularities of $a_r(X, Z)$ is relatively simple. If $a_r(X, Z)$ is singular at a point, say (X_0, Z_0), then it is singular at every point of the plane $Z=Z_0$, as follows from (4.3). That is, while in general the singularities of an analytic function of two complex variables lie on arbitrary analytic surfaces, the functions under consideration may be singular only on certain planes $Z=\text{const}$.

Lemma 3. *If the function $F(Z, Z^*)$ in (2.1) is entire and if $a_r(X, Z)$, $r>0$, is singular at the points of a plane $Z=Z_0$ lying in a simply connected domain $\mathfrak{D}$ containing the origin, then $a_0(X, Z)$ is singular at every point (X, Z_0) in $\mathfrak{D}$.*

Proof. Since the coefficient $F(Z, Z^*)$ in (2.1) is assumed to be an entire function, all the functions $c_p(Z)$ in the differential equations (4.4) are entire. Consequently, if $a_r(X, Z)$ is singular on the plane $Z=Z_0$, at least one of the functions $a_s(X, Z)$, $r-1 \geqq s \geqq 0$, appearing in (4.4) must be singular on this plane. By repeating this argument the statement follows.

Combining Theorem 1 and Lemma 3, we obtain the following result.

Theorem 4. *Let the coefficient $F(Z, Z^*)$ of the partial differential equation (2.1) be an entire function. If a solution $w(X, Z, Z^*)$ (cf. (3.4)) of (2.1) is regular for all values of Z in a simply connected domain $\mathfrak{D}$ containing $Z=0$, then the corresponding functions $a_p(X, Z)$, $p=1, 2, \ldots$, defined by (4.3), are regular for all values $Z \in \mathfrak{D}$ and $|X| < \infty$.*

6. The domain of regularity of solutions of (2.1)
in terms of arbitrary subsequences (k_{nmr}), $r > 0$ and fixed

The properties of the functions $a_1(X, Z)$, $a_2(X, Z)$, ... stated in Theorem 4 result from the regularity of the corresponding solution $w(X, Z, Z^*)$ of (2.1) in a certain domain. Unfortunately the condition that these functions have the above property is only necessary but not sufficient for the regularity of $w(X, Z, Z^*)$ in that domain. Indeed, let us consider the simplest of the equations (4.4), *i.e.*

$$\frac{\partial a_1}{\partial Z} = \frac{\partial^2 a_0}{\partial X^2} + c_0(Z)\, a_0. \tag{6.1}$$

From (3.1a), (3.5), and (4.3) we obtain the representation

$$a_0(X, Z) = \sum_{n=0}^{N} X^n f_n(Z). \tag{6.2}$$

Hence if $a_0(X, Z)$ has a plane of poles resulting from $f_0(Z)$ or $f_1(Z)$ and if this plane coincides with a plane of zeros of $c_0(Z)$ of sufficiently high order, then $a_1(X, Z)$ is regular at the points of this plane. By considering the other equations (4.4) in a similar manner we thus obtain the following intermediate result.

Lemma 5. *Suppose that the coefficient* $F(Z, Z^*)$ *of the partial differential equation* (2.1) *is an entire function. Let* $Z = Z_0$ *be a plane of poles of order* s *of* $a_0(X, Z)$ *resulting from the associated functions* $f_0(Z)$ *or* $f_1(Z)$. *Let* $Z = Z_0$ *be a zero of order* $m_j \geqq s$ *of* $c_j(Z)$, $j = 0, 1, \ldots, r-1$. *Then the functions* $a_1(X, Z), \ldots, a_r(X, Z)$ *are regular on the plane* $Z = Z_0$.

The coincidence of poles of $a_p(X, Z)$ with zeros of $c_i(Z)$ is the only reason that a plane of poles of $a_0(X, Z)$ might correspond to a plane of regular points of $a_p(X, Z)$, $p > 0$. This can be seen from the various equations (4.4). If $a_0(X, Z)$ has a plane of poles of the order s, say the plane $Z = Z_0$, and if $c_0(Z_0) \neq 0$ then $a_1(X, Z)$ has at $Z = Z_0$ a plane of poles of the order $s-1$ (or a plane of logarithmic singularities if $s = 1$), as follows from (6.1). In a similar way we find from the equation (4.4), $p = 2$, that if $a_0(X, Z)$ has a plane of poles $Z = Z_0$ and $c_0(Z_0) \neq 0$, $c_1(Z_0) \neq 0$ then $a_2(X, Z)$ is singular at $Z = Z_0$. Indeed, the functions involved in that equation have singularities of different orders which cannot cancel each other. Similar arguments hold with respect to essential singularities. The other functions $a_p(X, Z)$ can be treated in a similar way. Using Theorem 1 we thus obtain

Theorem 6. *Suppose that the coefficient* $F(Z, Z^*)$ *of* (2.1) *is an entire function. Let* $w(X, Z, Z^*)$ *be a solution of* (2.1) *which can be represented in the form* (3.4). *Suppose that in this representation the coefficients* k_{nmr}, $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, r > 0$ *and fixed, are such that* $a_r(X, Z)$ (cf. (4.3)) *is regular in a simply-connected domain* $\mathcal{D}$ *which includes the origin but no zeros of th functions* $c_j(Z)$, $j = 0, 1, \ldots, r-1$. *Then* $w(X, Z, Z^*)$ *is regular in* $\mathcal{D} \times (|X| < \infty)$.

7. Singularities of solutions of (2.1) in terms of arbitrary subsequences of coefficients in the corresponding power series development

By means of Theorem 1 various results on analytic functions can be "translated" into theorems on solutions of (2.1). These theorems involve only the subsequences (k_{nm0}), $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots$, of the coefficients in the series development (3.4). *Cf.* BERGMAN [7]. This fact is a consequence of the relation (3.2a) between the solutions $w(X, Z, Z^*)$ of (2.1) and their associated functions $f_0(Z), \ldots, f_N(Z)$. We shall now show how to apply the results in the previous sections to obtain more general theorems involving arbitrary subsequences (k_{nmr}), $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots$; $r > 0$ and fixed, of the coefficients in (3.4). It suffices to outline this procedure by means of some illustrative examples; the basic idea of deriving such theorems on solutions of (2.1) from theorems on analytic functions is similar in each particular case.

Let us generalize a theorem of BOREL on rational analytic functions.

Theorem 7. *Let* $w(X, Z, Z^*)$ *be a solution of* (2.1) *which can be represented in the form* (3.4). *Suppose that the function* $F(Z, Z^*)$ *in* (2.1) *is entire, and that the functions* $c_j(Z)$, $j = 0, 1, \ldots, r-1$, (cf. (4.1), (4.2)) *are different from zero in a simply connected domain* $\mathcal{B}$ *of the* Z-*plane which includes the origin. Then* $w(X, Z, Z^*)$ *has in* $\mathcal{B} \times (|X| < \infty)$ *at most finitely many planes of singularities (which are planes of poles of order not exceeding* $s + r$ *and (for all* $Z^* \neq 0$) *logarithmic branch planes) if and only if all but at most a finite number of the determinants*

$$D_{nmr} = \left| T_{nmr}^{\langle 1 \rangle}\, T_{nmr}^{\langle 2 \rangle} \ldots T_{nmr}^{\langle s+1 \rangle} \right|, \tag{7.1}$$

$$n = 0, 1, \ldots, N, \qquad m = 0, 1, \ldots, \qquad r > 0 \textit{ and fixed,}$$

with column vectors

$$T^{\langle j\rangle}_{nmr} = \{k_{n,\,m+j-1,\,r}\, k_{n,\,m+j,\,r} \cdots k_{n,\,m+j+s-1,\,r}\}$$

vanish.

Proof. The coefficients k_{nmp} appearing in the determinants (7.1) are just those which figure in the representation (4.3) of the function $a_r(X,Z)$, $r>0$ and fixed. We may represent $a_r(X,Z)$ in the form

$$a_r(X,Z) = \sum_{n=0}^{N} X^n h_n(Z) \tag{7.2}$$

where

$$h_n(Z) = \sum_{m=0}^{\infty} k_{nmr} Z^m, \tag{7.3}$$

$$n = 0, 1, \ldots, N, \qquad r > 0 \text{ and fixed.}$$

For each fixed value $n = \nu$, $0 \leqq \nu \leqq N$, the condition involving the determinant (7.1) is necessary and sufficient in order that $h_\nu(Z)$ be a rational function which has poles of order not exceeding s. *Cf.* E. Borel [*11*]. Hence the only singularities of $a_r(X,Z)$ are the planes of poles of order not exceeding s which correspond to the poles of the functions $h_n(Z)$, $n = 0, 1, \ldots, N$. In consequence of Lemma 3 the function $a_0(X,Z)$ is singular on these planes. Since the domain $\mathscr{B}$ does not include zeros of the functions $c_j(Z)$, $j = 0, 1, \ldots, r-1$, these planes of singularities are the only ones of this function in $\mathscr{B}$, which follows from Theorem 6. The nature of the singularities of $a_0(X,Z)$ can be determined by means of (4.4). One finds that the planes of singularities of $a_0(X,Z)$ are planes of poles of orders less than or equal to $s+r$. In consequence of (3.2a) these singularities correspond to poles of the associated functions $f_n(Z)$, $n = 0, 1, \ldots, N$, of the solution $w(X,Z,Z^*)$ under consideration. Since the coefficient $F(Z,Z^*)$ of (2.1) is assumed to be an entire function, all the functions $q_{n\beta p}(Z,Z^*)$ occurring in the representation (2.4) are entire. The integrals in (2.4) can be transformed into single integrals. By means of repeated integration by parts we obtain

$$\int_0^Z \int_0^{Z_1} \cdots \int_0^{Z_{p-1}} f_n(Z_p)\, dZ_{p-1} \ldots dZ_1 = \sum_{j=0}^{p-1} b_{jp} Z^{p-j-1} \int_0^Z t^j f_n(t)\, dt \tag{7.4}$$

where b_{jp}, $j = 0, 1, \ldots, p-1$, are constants; *cf.* Bergman [7]. If we substitute (7.4) into (2.4) and use the above result on the singularities of $a_0(X,Z)$, the statement on the type of singularities of $w(X,Z,Z^*)$ in $\mathscr{B}$ follows.

8. Singularities of real solutions on the boundary of the domain of convergence of their power series development

Let us consider another application of the results in the preceding sections of this paper and generalize a theorem by Bergman [7].

Theorem 8. *Let*

$$\begin{gathered} W(X,Z,Z^*) = \sum_{n=0}^{N} W_n(X,Z,Z^*; f_n) = \sum_{n=0}^{N} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} K_{nmp} X^n Z^m Z^{*p}, \\ K_{nmp} = \overline{K}_{npm}, \end{gathered} \tag{8.1}$$

be a real solution of (2.1) *whose associated functions* $f_n(Z)$, $n=0, 1, \ldots, N$, *are real at* $Z=0$; cf. (3.3) *and* (2.5). *Let* $F(Z, Z^*)$ *be entire*; cf. (2.1). *Suppose that*

$$\limsup_{m\to\infty} |K_{nmr}|^{\frac{1}{m}} = \varrho = \text{const}, \qquad n=0, 1, \ldots, N,$$

for an arbitrary fixed value $r>0$. *Suppose further that* $c_j(Z) \neq 0$ *for all values of* $|Z| \leqq \varrho$ *and* $j=0, 1, \ldots, r-1$. *Then* $W(X, Z, Z^*)$ *is singular on* $Z=Z_n=\varrho\,\mathrm{e}^{i\alpha_n}$ *if and only if*

$$\limsup_{s\to\infty} \left(\left| \sum_{\sigma=0}^{s} \binom{s}{\sigma} K_{n\sigma r} Z_n^{\sigma} \right| \right)^{\frac{1}{s}} = 2. \tag{8.2}$$

Proof. We introduce the functions

$$A_\lambda(X, Z) = \sum_{n=0}^{N} \sum_{m=0}^{\infty} K_{nm\lambda} X^n Z^m, \qquad \lambda = 0, 1, \ldots. \tag{8.3}$$

Then

$$W(X, Z, Z^*) = \sum_{\lambda=0}^{\infty} A_\lambda(X, Z) Z^{*\lambda}.$$

By substituting this representation into (2.1) we see that the functions $A_\lambda(X, Z)$ are related by a system of differential equations similar to (4.4). From this system it follows that the results of the preceding sections are also valid for real solutions of (2.1) for which each $f_n(0)$, $n=0, 1, \ldots, N$, is real. Using these results the above statement is a consequence of the following lemma (*cf.* BIEBERBACH [*12*], p. 35): Let

$$f(z) = \sum_{s=0}^{\infty} k_s z^s$$

be an analytic function regular at the origin. Then the point $z=z_0=\varrho\,\mathrm{e}^{i\alpha}$ on the circle of convergence $|z|=\varrho$, $\varrho = \limsup_{s\to\infty} |k_s|^{\frac{1}{s}}$ is a singular point of $f(z)$ if and only if

$$\limsup_{s\to\infty} |H_s|^{\frac{1}{s}} = 2$$

where

$$H_s = \sum_{\sigma=0}^{s} \binom{s}{\sigma} k_\sigma \varrho^\sigma \mathrm{e}^{i\sigma\alpha}.$$

Indeed, from this lemma it follows that the above assumption on the sequences (K_{nmr}), $n=0, 1, \ldots, N$, $m=0, 1, \ldots$, $r>0$ and fixed, is the necessary and sufficient condition for the function $A_r(X, Z)$ to be regular for $|Z|<\varrho$, $|X|<\infty$ and singular on $Z=Z_n$, $n=0, 1, \ldots, N$, $|X|<\infty$. Hence, by Lemma 3, the function $A_0(X, Z)$ is singular at those points. Because of the assumptions on $c_j(Z)$, $j=0, 1, \ldots, r-1$, it follows from Theorem 6 that these are the only singularities of $A_0(X, Z)$ in $|Z| \leqq \varrho$, $|X|<\infty$. From this, (3.3), and (3.2a) the statement follows.

9. Final remark

So far we have assumed that the coefficient $F(Z, Z^*)$ of the partial differential equation (2.1) is an entire function. This is essential to the proofs of all the preceding results. Indeed, the situation changes if we permit $F(Z, Z^*)$ to have singularities. We confine ourselves to a brief remark in this direction.

Suppose that $F(Z,0)=c_0(Z)$ $\big(cf.\ (4.2)\big)$ has a pole at a point $Z=Z_0$. Let $w(X,Z,Z^*)$ be a solution of (2.1) which can be represented in the form (3.4) and has the property that $w(X,Z_0,0)=a_0(X,Z_0)\neq 0$. Then the corresponding function $a_1(X,Z)$ is singular on $Z=Z_0$. However, if all the associated functions $f_0(Z),\ldots,f_N(Z)$ of $w(X,Z,Z^*)$ have zeros of sufficiently high order at $Z=Z_0$, then $a_1(X,Z)$ is regular in the plane $Z=Z_0$. This and similar statements regarding the functions $a_p(X,Z)$, $p=1,2,\ldots$, follow from the differential equations (4.4).

Lemma 3 will no longer be true if $F(Z,Z^*)$ is not entire. A plane of singularities of $a_p(X,Z)$ may now result from either $a_0(X,Z)$ or $F(Z,Z^*)$. The situation with respect to other results in this paper is similar.

Theorem 9. *Suppose that the coefficient F in* (2.1) *is a function of Z only and has a singularity at a point $Z=Z_0$. Then for every solution $w(X,Z,Z^*)$ of* (2.1) *which can be represented in the form* (3.4) *and which has the property that $w(X,Z_0,0)\neq 0$, the corresponding functions $a_p(X,Z)$, $p=1,2,\ldots$, are singular at each point of the plane $Z=Z_0$ in the domain of existence of the solution.*

This result follows from (4.4).

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(Received June 9, 1958)

Sonic Limit Singularities. Part I: General Theory

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Communicated by Hilda Geiringer

The sonic limit line was first identified by Hilda Geiringer [*1*]. In the same paper she also gave a classification of limit singularities for $M=1$ which was later adopted by the present authors [*2*]. In analogy with the treatment of limit singularities in the supersonic region [*1*], this classification was based on the vanishing of ψ_ξ and ψ_η at points in the hodograph, where ψ is the streamfunction and ξ and η are the characteristic variables.

Further investigation has shown that this classification is not adequate since the ξ- and η-directions become indistinct at the sonic circle and consequently the vanishing of ψ_ξ and ψ_η at a sonic point may depend on the path of approach in the hodograph plane. In the present paper we suggest a new classification (Sections 3—6) which is based on the vanishing of ψ_q and ψ_ϑ, where q and ϑ are the hodograph variables, and we discuss the geometry in each case without, however, reproducing the results derived in [*2*] and [*5*]. To this end we first show (Section 2), under the assumption that ψ is a regular function of the hodograph variables, that a sonic limit point cannot be an isolated limit point, *i.e.* it must either lie on a sonic limit line or on a supersonic limit line. Examples of flows exhibiting these singularities will be presented in Part II*.

1. Introduction

First we recall briefly some of the basic equations stated in [*2*].

In the hodograph variables q, ϑ the streamfunction ψ satisfies Chaplygin's equation

$$\psi_{qq} - \frac{(M^2-1)}{q^2}\,\psi_{\vartheta\vartheta} + \frac{(M^2+1)}{q}\,\psi_q = 0 \tag{1}$$

where

$$M^2 = \frac{q^2}{a^2} = \frac{2}{\gamma-1}\cdot\frac{q^2}{1-q^2}\,.$$

For any non-constant solution $\psi(q,\vartheta)$ of this equation the physical coordinates $x(q,\vartheta)$ and $y(q,\vartheta)$ are determined by

$$\begin{aligned} dx &= \frac{1}{\varrho q}\left\{\left[-\sin\vartheta\,\psi_q + \frac{M^2-1}{q}\cos\vartheta\,\psi_\vartheta\right]dq + \left[q\cos\vartheta\,\psi_q - \sin\vartheta\,\psi_\vartheta\right]d\vartheta\right\}\\ dy &= \frac{1}{\varrho q}\left\{\left[\cos\vartheta\,\psi_q + \frac{M^2-1}{q}\sin\vartheta\,\psi_\vartheta\right]dq + \left[q\sin\vartheta\,\psi_q + \cos\vartheta\,\psi_\vartheta\right]d\vartheta\right\}\end{aligned} \tag{2}$$

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* Geiringer [*6*] has pointed out the occurrence of corresponding singularities in plasticity theory, where the present results should have counterparts.

where ϱ is the density,

$$\varrho = (1 - q^2)^{\frac{1}{\gamma-1}}.$$

The Jacobian of the transformation from the hodograph to the physical plane is

$$I = \frac{\partial(x, y)}{\partial(q, \vartheta)} = -\frac{1}{\varrho^2 q^3}\left[q^2 \psi_q^2 - (M^2 - 1)\,\psi_\vartheta^2\right].$$

In the supersonic region ($M > 1$) it is possible to introduce the characteristic coordinates ξ, η of (1) by

$$2\xi = \int^q \frac{\sqrt{M^2-1}}{q}\,dq - \vartheta, \quad 2\eta = \int^q \frac{\sqrt{M^2-1}}{q}\,dq + \vartheta$$

and write the Jacobian in the form

$$I = -\frac{1}{\varrho^2 q^3}(M^2 - 1)\,\psi_\xi \psi_\eta.$$

Breakdown of the transformation will occur at points in the hodograph plane where $I = 0$, and their images are called *limit points*. Curves consisting of such image points are called *limit lines*.

Clearly when $M > 1$ the vanishing of I implies that either ψ_ξ, or ψ_η, or both vanish. A curve in the physical plane along which $\psi_\xi = 0$, $\psi_\eta \neq 0$ is called a *limit line* $\mathscr{L}_1$, and one along which $\psi_\xi \neq 0$, $\psi_\eta = 0$ is called a *limit line* $\mathscr{L}_2$.* The point of intersection of two such curves is called a *double limit point*.

2. Non-isolation of the sonic limit point

A question which immediately suggests itself is whether limit singularities at sonic points can be isolated or must lie on a line of singularities. It is well known that limit points in the subsonic region must be isolated, whereas ordinary and double limit points occurring in the supersonic region must lie on supersonic limit lines. Under the assumption that ψ is regular in the hodograph plane, the main result (Theorems 1—4) proved in this section is that every sonic limit point must lie on some kind of limit *line*. This line either consists exclusively of sonic limit points — and is then called a *sonic limit line* — or it contains supersonic limit points, in which case the point under consideration is a *sonic point on a supersonic limit line*. An investigation of the geometry in the neighborhood of these points (which differs in several respects from that for supersonic limit points) will be the subject of Sections 3—6.

Throughout the present discussion we shall assume that ψ is a regular function of q and ϑ. It then follows that the Jacobian $I = -[q^2 \psi_q^2 - (M^2 - 1)\,\psi_\vartheta^2]/\varrho^2 q^3$ vanishes for $M = 1$ $(q = q_t)$ if and only if $\psi_q = 0$.

A point at which $M = 1$, $\psi_q = 0$, $\psi_\vartheta \neq 0$ will be called an *ordinary sonic limit point*, and one at which $M = 1$, $\psi_q = 0$, $\psi_\vartheta = 0$ will be called a *higher-order sonic limit point*. It follows immediately from Chaplygin's equation that at a sonic limit point $\psi_{qq} = 0$.

A line along which $M = 1$, $\psi_q = 0$ is called a *sonic limit line*, $\mathscr{L}_t$. Along such a line $\psi_q = \psi_{q\vartheta} = \psi_{q\vartheta\vartheta} = \psi_{q\vartheta\vartheta\vartheta} = \cdots = 0$, and $\psi_{qq} = \psi_{qq\vartheta} = \psi_{qq\vartheta\vartheta} = \psi_{qq\vartheta\vartheta\vartheta} = \cdots = 0$.

---

* Their images in the hodograph plane will be denoted by l_1 and l_2, respectively.

Theorem 1. *Every ordinary sonic limit point lies on a sonic or supersonic limit line.*

Proof. At an ordinary sonic limit point $M=1$, $\psi_q=0$, $\psi_\vartheta\neq 0$. Hence with $F=-\varrho^2 qI$:

$$F\equiv\psi_q^2-\left(\frac{1}{a^2}-\frac{1}{q^2}\right)\psi_\vartheta^2=0$$

$$F_q=2\psi_q\psi_{qq}-2\psi_\vartheta\psi_{\vartheta q}\left(\frac{1}{a^2}-\frac{1}{q^2}\right)-\psi_\vartheta^2\frac{d}{dq}\left(\frac{1}{a^2}-\frac{1}{q^2}\right)=-\frac{\gamma+1}{q_t^3}\psi_\vartheta^2\neq 0$$

$$F_\vartheta=2\psi_q\psi_{q\vartheta}-2\psi_\vartheta\psi_{\vartheta\vartheta}\left(\frac{1}{a^2}-\frac{1}{q^2}\right)=0,$$

and by the implicit function theorem there exists a curve $q=f(\vartheta)$ through the point[*] on which $F\equiv 0$ with the direction

$$\frac{dq}{d\vartheta}=-\frac{F_\vartheta}{F_q}=0,\qquad q=\text{const}.$$

For higher-order sonic limit points the non-isolation cannot be deduced from the implicit function theorem since $F_q=F_\vartheta=0$ (and if $F_{q\vartheta}=0$ then $F_{q\vartheta}^2-F_{qq}F_{\vartheta\vartheta}=0$). To handle this case we consider the sign of $F^\pm=\psi_q\pm\left(\sqrt{M^2-1}/q\right)\psi_\vartheta$ on certain families of curves in the hodograph.

Lemma. *If at a sonic point all derivatives of ψ with respect to q and ϑ vanish up to and including the n^{th} order, then all derivatives of order $n+1$ vanish, except possibly $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}$ and $\psi_{\vartheta\underbrace{\vartheta\ldots\vartheta}_{n}}$.*

Proof. This follows by successive differentiation of Chaplygin's equation (1).

Thus the discussion of the general higher-order sonic limit point where all derivatives of ψ up to and including the n^{th} order vanish can be broken down into the following two cases:

(a) The first (in the order of decreasing q-subscripts) non-vanishing derivative of ψ of $(n+1)^{\text{st}}$ order is $\psi_{\underbrace{\vartheta\ldots\vartheta}_{1+n}}$.

(b) The first non-vanishing derivative of ψ is $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{1+n}}$.

In the second case, we have to consider n odd and n even separately.

Theorem 2. *In case* (a) *the sonic limit point is not isolated.*

Proof. Let $\frac{1}{n!}\psi_{\vartheta\underbrace{\vartheta\ldots\vartheta}_{n}}=\bar{k}\neq 0$. Then expanding in a neighborhood of the point[**],

$$\psi_q=O(\delta q,\delta\vartheta)^{n+1},\qquad\qquad \psi_q^2=O(\delta q,\delta\vartheta)^{2n+2}$$

$$\psi_\vartheta=\bar{k}(\delta\vartheta)^n+O(\delta q,\delta\vartheta)^{n+1},\qquad \psi_\vartheta^2=\bar{k}^2(\delta\vartheta)^{2n}+O(\delta q,\delta\vartheta)^{2n+1}$$

$$\frac{M^2-1}{q^2}=\frac{\gamma+1}{q_t^3}\delta q+O(\delta q)^2,$$

---

[*] Strictly speaking we should write "image point in the hodograph". The abbreviation is used whenever the meaning is clear.

[**] $O(\delta q,\delta\vartheta)^n$ denotes a regular function of δq and $\delta\vartheta$ with terms $(\delta q)^k(\delta\vartheta)^l$ where $k+l\geq n$.

so that
$$F = \psi_q^2 - \frac{M^2-1}{q^2}\psi_\vartheta^2 = -\frac{\gamma+1}{q_t^3}\bar{k}^2(\delta q)\,(\delta\vartheta)^{2n} + O(\delta q, \delta\vartheta)^{2n+2}.$$

Thus on the straight line $\delta q = m\,\delta\vartheta$ the principal part of F behaves as follows
$$\frac{F}{(\delta\vartheta)^{2n+1}} = -\frac{\gamma+1}{q_t^3}\bar{k}^2 m\,.$$

Thus for sufficiently small $\delta\vartheta \neq 0$ we have that $F \gtrless 0$ according as $\delta q = m\,\delta\vartheta \lessgtr 0$. This means that F takes on positive and negative values for various choices of the parameter m as shown in Figure 1.

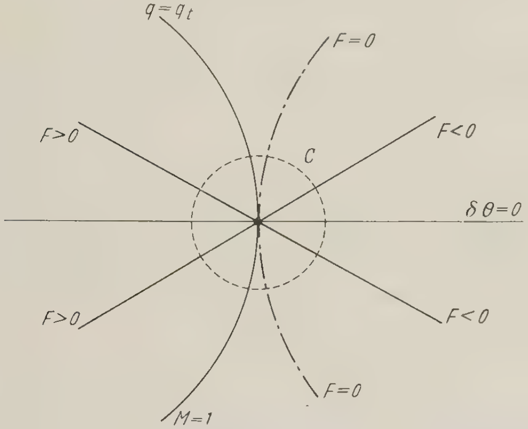

Fig. 1. The sign of F in case (a)

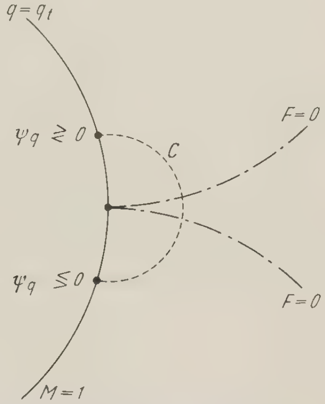

Fig. 2. Vanishing of F in case (b) with n odd

Now F is a continuous function on any small circle C about the point, and hence there exist points on C such that $F=0$. Thus F must vanish on a curve through the point, and this curve cannot lie in the subsonic region, since limit points are isolated there.

Theorem 3. *In case* (b) *with n odd the sonic limit point is a sonic point on a supersonic limit line.*

Proof. The fact that the first non-vanishing derivative of ψ_q with respect to ϑ is of odd order implies by Taylor's theorem that ψ_q changes sign on $q=q_t$. This implies that the continuous function $F^+ = \psi_q + \left(\sqrt{M^2-1}/q\right)\psi_\vartheta$ tends to values with opposite signs at the ends of any sufficiently small "semicircle" C about the point on the supersonic side. (See Figure 2.) Hence $F^+ = 0$ for some supersonic point on this semicircle. The same result holds for F^-.

Lemma. *Consider case* (b) *with n even and the derivatives of ψ of order $1+n+k$ at the point. Then for $k = 1, 2, \ldots, \frac{1}{2}n$*
$$(\psi_q)_{\underbrace{q\ldots q}_{n+k-p}\,\underbrace{\vartheta\ldots\vartheta}_{p}} = \begin{cases} 0, & p < n-2k, \\ K(k)\,\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}, & p = n-2k, \end{cases}$$
and for $k = 1, 2, \ldots, \frac{1}{2}n - 1$,
$$(\psi_\vartheta)_{\underbrace{q\ldots q}_{n+k-p}\,\underbrace{\vartheta\ldots\vartheta}_{p}} = \begin{cases} 0, & p < n-2k-1, \\ K(k)\,\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}, & p = n-2k-1, \end{cases}$$

where, apart from k, $K(k)$ depends only on the first derivative of $(1-M^2)/q^2$ at $M=1$. That is, the first nonvanishing derivative of order $1+n+k$ in the order of decreasing q-subscripts contains $1+3k$ differentiations with respect to q and is a multiple of $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}$ only.

Proof. Successive differentiation of Chaplygin's equation (1) shows that all derivatives of ψ of order $1+n+k$ which involve more than $1+3k$ differentiations with respect to q can be expressed in terms of derivatives of order n and lower, and derivatives of order $1+n$ other than $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}$ and $\psi_{\vartheta\underbrace{\vartheta\ldots\vartheta}_{n}}$. By the previous lemma, then, these derivatives of order $1+n+k$ must vanish at the point.

Similarly, we see that the derivative of order $1+n+k$ which involves precisely $1+3k$ differentiations with respect to q can be expressed in terms of $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}$ as indicated.

It is important to note that the same results (including the function $K(k)$), are obtained if in Chaplygin's equation the coefficient $(1-M^2)/q^2$ is replaced by the first term in its Taylor expansion at $M=1$, and the coefficient of ψ_q is replaced by zero. Thus instead of Chaplygin's equation we may consider Tricomi's equation

$$(3) \qquad \psi_{yy} - y\,\psi_{xx} = 0, \qquad x=\vartheta, \qquad y = \frac{\gamma+1}{q_t^3}\,\delta q,$$

whenever we are concerned solely with the derivatives indicated in the lemma.

Theorem 4. *In case* (b) *with n even the sonic limit point is a sonic point on a supersonic limit line.*

Proof. From the lemma, the sum of the leading terms (one for each k) in the Taylor expansion of $F^{\pm} = \psi_q \pm (\sqrt{M^2-1/q})\,\psi_\vartheta$ on the semi-cubical parabola $(\sqrt{(\gamma+1)/q_t^3\,\delta q})^{\frac{3}{2}} - \frac{3}{2}p\,\delta\vartheta$ is a multiple of $\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}$.* Now, from the remark above

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* From the lemma, the principal parts of the Taylor expansions of ψ_q and ψ_ϑ about the point reduce to

$$\psi_q = \frac{1}{n!}\,\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}(\delta\vartheta)^n + \sum_{k=1}^{k-\frac{1}{2}n} \frac{1}{(n+k)!}\binom{n+k}{3k} K(k)\,\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}(\delta q)^{3k}(\delta\vartheta)^{n-2k},$$

$$\psi_\vartheta = \frac{1}{n!}\binom{n}{1}\psi_{\vartheta q\underbrace{\vartheta\ldots\vartheta}_{n}}(\delta q)(\delta\vartheta)^{n-1} +$$
$$+ \sum_{k=1}^{k=\frac{1}{2}n-1} \frac{1}{(n+k)!}\binom{n+k}{3k+1} K(k)\,\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}(\delta q)^{3k+1}(\delta\vartheta)^{n-2k-1}.$$

Hence the principal part of $F^{\pm}$ on $(\sqrt[3]{(\gamma+1)/q_t^3\,\delta q})^{\frac{3}{2}} = \frac{3}{2}p\,\delta\vartheta$ is

$$F^{\pm} = \left\{\left[\frac{1}{n!} + \sum_{k=1}^{k=\frac{1}{2}n} \frac{1}{(n+k)!}\binom{n+k}{3k}[(3k-1)(3k-4)\ldots 8\cdot 5\cdot 2]\left(\frac{3}{2}p\right)^{2k}\right] \pm\right.$$
$$\left.\pm\left[\frac{1}{(n-1)!}\left(\frac{3}{2}p\right) + \sum_{k=1}^{k=\frac{1}{2}n-1} \frac{1}{(n+k)!}\binom{n+k}{3k+1}[(3k-1)\ldots 8\cdot 5\cdot 2]\left(\frac{3}{2}p\right)^{2k+1}\right]\right\}\psi_{q\underbrace{\vartheta\ldots\vartheta}_{n}}(\delta\vartheta)^n.$$

However, it is difficult to discuss changes in sign of $F^{\pm}$ from this representation.

it is clear that the same leading terms result if Tricomi's equation (3) is used and $G^{\pm} = \psi_y \pm y^{\frac{1}{2}} \psi_x$ is expanded on $y^{\frac{3}{2}} = \frac{3}{2} p\, x$. Moreover, these leading terms remain unaltered if we construct any other solution of Tricomi's equation which gives the same $\psi_{y \underbrace{x \ldots x}_{n}}$ at the point considered and has the same earlier derivatives equal to zero there.

Now it is well known [3] that the general solution of Tricomi's equation is given by

$$\psi = \frac{\Gamma(\frac{1}{3})}{\Gamma^2(\frac{1}{6})} \int_0^1 \tau\left[x + \frac{2}{3} y^{\frac{3}{2}}(2t-1)\right] t^{\;\frac{5}{6}}(1-t)^{-\frac{5}{6}}\, dt + $$
$$+ \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} y \int_0^1 \nu\left[x + \frac{2}{3} y^{\frac{3}{2}}(2t-1)\right] t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt$$

where

$$\tau(x) = \psi(x,0), \qquad \nu(x) = \left[\frac{\partial \psi}{\partial y}\right]_{y=0},$$

and in view of the above results we select the solution determined by the Cauchy data

$$\tau(x) = \frac{1}{n!} \psi_{y \underbrace{x \ldots x}_{n}} x^n = \alpha\, x^n, \qquad \nu(x) \equiv 0,$$

namely

$$\psi = \alpha \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} y \int_0^1 \left[x + \frac{2}{3} y^{\frac{3}{2}}(2t-1)\right]^n t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt.$$

Forming $G^{\pm}$, we have

$$G^{\pm} = \alpha \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} \int_0^1 \left[x + \frac{2}{3} y^{\frac{3}{2}}(2t-1)\right]^{n-1} \times$$
$$\times \left\{x + \frac{2}{3} y^{\frac{3}{2}}(2t-1) + n\, y^{\frac{3}{2}}[2t - 1 \pm 1]\right\} t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt,$$

and on $y^{\frac{3}{2}} = \frac{3}{2} p\, x$

$$G^{\pm}(p) = \alpha\, x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} \int_0^1 [1 + p(2t-1)]^{n-1} \times$$
$$\times \left[1 + p\left(1 + \frac{3}{2} n\right)(2t-1) \pm \frac{3}{2} n\, p\right] t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt^{\star}.$$

We now show that $G^{\pm}(p)$ changes sign for certain values of p. Assume that $\alpha = \frac{1}{n!} \psi_{y \underbrace{x \ldots x}_{n}} > 0$. Then for $p = 0$ we have

$$G^{\pm}(0) = \alpha\, x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} \int_0^1 t^{-\frac{1}{6}}(1-t)^{\;\frac{1}{6}}\, dt - \alpha\, x^n > 0,$$

and for the characteristic $p = 1$ of the Tricomi equation

$$G^{+}(1) = \alpha\, x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} 2^{n-1} \int_0^1 [2 + 3nt]^{n-\frac{7}{6}} (1-t)^{-\frac{1}{6}}\, dt$$
$$= \alpha\, x^n \frac{\Gamma(\frac{5}{3})\, \Gamma(n+\frac{5}{6})}{\Gamma(\frac{5}{6})\, \Gamma(n+\frac{5}{3})} 2^{n\;1}(2 + 3n) > 0,$$

★ This integral is of course identical to the corresponding polynomial appearing in the previous footnote.

and

$$G^-(1) = \alpha x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} 2^{n-1} \int_0^1 [2t - 3n(1-t)]\, t^{n-\frac{7}{6}} (1-t)^{-\frac{1}{6}}\, dt$$

$$= -\alpha x^n \frac{\Gamma(\frac{5}{3})\, \Gamma(n-\frac{1}{6})}{\Gamma(\frac{5}{6})\, \Gamma(n+\frac{5}{3})} 2^{n-2} (2+3n) < 0.$$

Similarly we have for the other characteristic $p = -1$

$$G^\pm(-1) = \alpha x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} 2^{n-1} \int_0^1 \left[1 - \left(1 + \frac{3}{2} n\right)(2t-1) \mp \frac{3}{2} n\right] t^{-\frac{1}{6}} (1-t)^{n-\frac{7}{6}}\, dt.$$

Now letting $t = 1 - s$

$$G^\pm(-1) = \alpha x^n \frac{\Gamma(\frac{5}{3})}{\Gamma^2(\frac{5}{6})} 2^{n-1} \int_0^1 \left[1 + \left(1 + \frac{3}{2} n\right)(2s-1) \mp \frac{3}{2} n\right] (1-s)^{-\frac{1}{6}} s^{n-\frac{1}{6}}\, ds$$

so that $G^\pm(-1) = G^\mp(1)$ and hence $G^-(-1) > 0$ and $G^+(-1) < 0$.

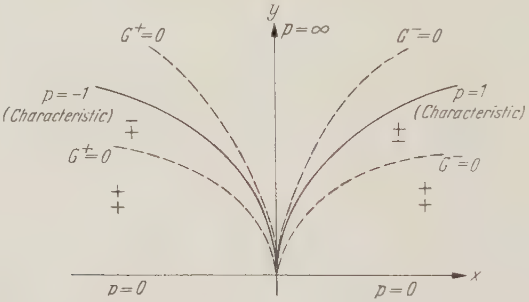


Fig. 3. The sign of $G\pm(F\pm)$; case (b) with n even

Thus on $y^{\frac{3}{2}} = \frac{3}{2} p x$ the quantity $G^\pm$ takes on positive and negative values in the neighborhood of the point as shown in Figure 3 and there must be at least two supersonic curves on which $G^+ = 0$ and two supersonic curves on which $G^- = 0$ entering the point.

3. Ordinary sonic limit point on a sonic limit line ($\psi_\vartheta \neq 0$)

(Examples of such points are found in the radial flow and in the flow discussed in [2].)

Theorem 5. *Neither an $\mathscr{L}_1$ nor an $\mathscr{L}_2$ can come into the point.*

Proof. The proof of this theorem is presented in [5] and depends on the fact that, due to the presence of the sonic limit line ($\psi_q \equiv 0$ on $q = q_t$), we have from Taylor's formula applied to ψ_q that $\psi_q = O(\delta q)$. Then $q\,\psi_q/\sqrt{M^2-1} = O(\delta q)^{\frac{1}{2}}$, so that $\psi_\xi = \left(q\,\psi_q/\sqrt{M^2-1}\right) - \psi_\vartheta$ tends to the value $-\psi_\vartheta$ at $q = q_t$ irrespective of the path of approach. But if the sonic point of an $\mathscr{L}_1$ (on which $\psi_\xi = 0$) is approached along the $\mathscr{L}_1$, ψ_ξ clearly tends to zero. Hence $\psi_\vartheta = 0$ if such a point belongs to a sonic limit line. A similar argument holds for the $\mathscr{L}_2$.

The geometry of the flow in the neighborhood of such a point was derived in [2]; see also [5].

Theorem 6. *If $\psi_{\vartheta\vartheta} \neq 0$, the streamline in the hodograph inflects at the image point on passing from the subsonic to the supersonic region.*

Proof. First we note that there is a streamline through the image point. This follows from the implicit function theorem since $\psi_\vartheta \neq 0$.

Differentiating Chaplygin's equation with respect to q and applying the conditions of an ordinary sonic limit point we have that $\psi_{qqq} \neq 0$ if (and only if) $\psi_{\vartheta\vartheta} \neq 0$. Hence using q as parameter on the streamline we find that

$$\frac{d\vartheta}{dq} = 0, \qquad \frac{d^2\vartheta}{dq^2} = 0 \qquad \frac{d^3\vartheta}{dq^3} = -\frac{\psi_{qqq}}{\psi_\vartheta} \neq 0.$$

Theorem 7. *Any curve in the physical plane without inflection can be a sonic limit line, the two flows which meet at it lying on the convex side.*

Proof. Let the curve be denoted by $y = f(x)$ and define the stream direction ϑ at each point by

$$\cot\vartheta = -f'(x). \tag{4}$$

Considering ϑ as a parameter on the curve: $x = g(\vartheta)$, $y = h(\vartheta)$, we now set

$$F(\vartheta) = -\operatorname{cosec}\vartheta\, g'(\vartheta) = \sec\vartheta\, h'(\vartheta), \tag{5}$$

and construct the hodograph solution $\psi(q, \vartheta)$ which satisfies the conditions

$$\psi_q = 0, \qquad \psi_\vartheta = \varrho_t q_t F(\vartheta) \quad \text{on} \quad q = q_t.$$

Then the corresponding flow in the physical plane clearly contains a sonic limit line, whose coordinates (see (2)) are given by

$$\begin{aligned} x &= -\int F(\vartheta) \sin\vartheta\, d\vartheta = g(\vartheta), \\ y &= \int F(\vartheta) \cos\vartheta\, d\vartheta = h(\vartheta), \end{aligned}$$

when the integration constants are suitably chosen. It is easily checked that the flow is unique, except for reversal of direction, since (4) must hold at a sonic limit line (see [5]).

Consider now the map of a radial element at the sonic line in the hodograph plane. By differentiating (2), we find $dx/dq = dy/dq = 0$ and

$$\begin{aligned} \frac{d^2x}{dq^2} &= \frac{(\gamma-1)\cos\vartheta}{\varrho_t q_t^5}\psi_\vartheta = \frac{(\gamma-1)}{q_t^4}\cos\vartheta\, F(\vartheta) = \frac{(\gamma-1)}{q_t^4} h'(\vartheta), \\ \frac{d^2y}{dq^2} &= \frac{(\gamma-1)}{q_t^4}\sin\vartheta\, F(\vartheta) = -\frac{(\gamma-1)}{q_t^4} g'(\vartheta), \end{aligned}$$

where (5) has been used (*cf.* Eq. (8) on p. 232 of [2]). Hence whether the element points into the subsonic or supersonic region, it maps into an element in the direction $(h'(\vartheta), -g'(\vartheta))$ perpendicular to the sonic limit line. But according to (4): $f''(x)\, g'(\vartheta) = \operatorname{cosec}^2\vartheta > 0$, so that g' has the same sign as f''. Hence the image element points towards the convex side of the curve.

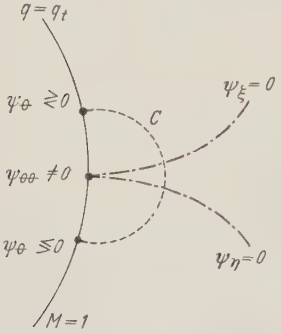
4. Higher-order sonic limit point on a sonic limit line ($\psi_\vartheta = 0$)

Any point at which an $\mathcal{L}_1$ or $\mathcal{L}_2$ meets a sonic limit line is such a point (see Theorem 5), and any higher-order sonic limit point is necessarily isolated from others of the same kind (since otherwise $\psi(q, \vartheta) \equiv$ constant). In particular, it follows that neither an $\mathcal{L}_1$ nor an $\mathcal{L}_2$ can coincide with the sonic line, or in other words a sonic limit line cannot be an $\mathcal{L}_1$ or $\mathcal{L}_2$.

(i) *Higher-order sonic limit point such that* $\psi_{\vartheta\vartheta} \neq 0$. (An example of a flow with this type of singularity was constructed in [2]. Other examples will appear in Part II, Section 5.)

Theorem 8. *An odd number of* $\mathcal{L}_1$ *and an odd number of* $\mathcal{L}_2$ *enter the point.*

Proof. As in the proof of Theorem 5 we have, since $\psi_q = 0$ along the sonic circle, that $\psi_\xi = \left(q\,\psi_q/\sqrt{M^2-1}\right) - \psi_\vartheta$ tends to the value $-\psi_\vartheta$ at $q = q_t$. Now the function $\psi_\xi(q, \vartheta)$ is continuous on any sufficiently small supersonic "semicircle" C about the point and tends to values with opposite signs at the ends (see Figure 4). Hence there exist an odd number of points on C at which $\psi_\xi = 0$. Similarly for $\psi_\eta = 0$.



$q=q_t$
$\psi_\vartheta \gtrless 0$
C
$\psi_\xi = 0$
$\psi_{\vartheta\vartheta} \neq 0$
$\psi_\vartheta \lessgtr 0$
$\psi_\eta = 0$
$M=1$

Fig. 4. Higher-order sonic limit point ($\psi_{\vartheta\vartheta} \neq 0$)

Theorem 9. *There is just one* $\mathcal{L}_1$ *and one* $\mathcal{L}_2$ *entering the point and their images* l_1, l_2 *together form a cusp of the first kind* in the radial direction.*

Proof. Making use of the conditions $\psi_q = \psi_{\vartheta\vartheta} = \psi_{qq} = \psi_{q\vartheta} = 0$, $\psi_{\vartheta\vartheta} \neq 0$ (note that the sonic limit line condition is embodied in $\psi_{q\vartheta} = 0$), and the fact that $\psi_{\vartheta\vartheta} \neq 0$ only if $\psi_{qqq} \neq 0$ (*cf.* Theorem 6), we have

$$F = F_q = F_\vartheta = F_{qq} = F_{q\vartheta} = F_{\vartheta\vartheta} = F_{qqq} = F_{qq\vartheta} = F_{\vartheta\vartheta\vartheta} = 0,$$

$$F_{q\vartheta\vartheta} = -2\psi_{\vartheta\vartheta}^2 \left(\frac{\gamma+1}{q_t^3}\right) \neq 0, \qquad F_{qqqq} = 6\psi_{qqq}^2 \neq 0.$$

Thus the leading terms in the Taylor expansion of the curve $F(q, \vartheta) = 0$ are

$$F = \frac{1}{3!}\left[3F_{q\vartheta\vartheta}(\delta q)(\delta\vartheta)^2\right] + \frac{1}{4!}\left[F_{qqqq}(\delta q)^4 + \cdots\right] + \cdots = 0.$$

Hence any branch enters the point either in the q-direction or in the ϑ-direction.

For any branch in the q-direction we may write $\delta q = x$, $\delta\vartheta = u x$, where u tends to zero with x, and in the x, u-plane this branch is determined by

$$G(x, u) = \frac{1}{x^3} F = \frac{1}{3!}\left[3F_{q\vartheta\vartheta} u^2\right] + \frac{1}{4!}\, x\left[F_{qqqq} + \cdots\right] + \cdots = 0.$$

However, since $\partial G/\partial x = F_{qqqq}/4! \neq 0$, there is a unique curve

$$x = f(u) = -12\frac{F_{q\vartheta\vartheta}}{F_{qqqq}}u^2 + \cdots$$

through the origin. Its image (cusp of first kind)

$$(\delta q)^3 = -12\frac{F_{q\vartheta\vartheta}}{F_{qqqq}}(\delta\vartheta)^2 + \cdots$$

---

* See [4], where the method of our proof is given.

in the hodograph plane therefore provides all branches (*viz.* two) of $F=0$ which enter in the radial direction.

Similarly we see that there is only one branch entering in each of the positive and negative ϑ-directions. These together form the sonic limit line. Thus the cusp must be formed by a unique l_1 and a unique l_2 (*cf.* Theorem 8).

Theorem 10. *The sonic limit line in the physical plane is cusped at the point.*

Proof. From the transformation equations (2) we have along a sonic limit line (using ϑ as a parameter)

$$\frac{dx}{d\vartheta}=\frac{1}{\varrho q}\left(-\sin\vartheta\,\psi_\vartheta\right), \qquad \frac{dy}{d\vartheta}=\frac{1}{\varrho q}\left(\cos\vartheta\,\psi_\vartheta\right)$$

and

$$\frac{d^2x}{d\vartheta^2}=\frac{1}{\varrho q}\left(-\cos\vartheta\,\psi_\vartheta-\sin\vartheta\,\psi_{\vartheta\vartheta}\right), \qquad \frac{d^2y}{d\vartheta^2}=\frac{1}{\varrho q}\left(-\sin\vartheta\,\psi_\vartheta+\cos\vartheta\,\psi_{\vartheta\vartheta}\right).$$

Hence we see that at the point both first derivatives, but not both second derivatives vanish.

Theorem 11. *The streamline $\psi(q,\vartheta)=c$ which goes through the image point in the hodograph plane has a cusp of the first kind there. The cusp tangent has the radial direction.*

Proof. Let $\overline{\psi}=\psi-c$. Then

$$\overline{\psi}=\overline{\psi}_q=\overline{\psi}_\vartheta=\overline{\psi}_{qq}=\overline{\psi}_{q\vartheta}=0, \qquad \overline{\psi}_{\vartheta\vartheta}\neq 0 \quad \text{and hence} \quad \overline{\psi}_{qqq}\neq 0$$

(*cf.* Theorem 6). The Taylor expansion of $\overline{\psi}$ is

$$\overline{\psi}=\frac{1}{2!}\left[\overline{\psi}_{\vartheta\vartheta}(\delta\vartheta)^2\right]+\frac{1}{3!}\left[\overline{\psi}_{qqq}(\delta q)^3+\cdots\right]+\cdots.$$

Thus according to [4] the curve $\overline{\psi}(q,\vartheta)$ forms a single cusp of the first kind with cusp tangent in the radial direction.

(ii) *Higher-order sonic limit point such that* $\psi_{\vartheta\vartheta}=0$. There exist cases where no supersonic limit line enters the point. A necessary condition for this (*cf.* Theorem 8) is that ψ_ϑ does not change sign on $q=q_t$. An example is presented in Part II, Section 7.

The sonic limit line need not be cusped. For from the proof of Theorem 10 we see that $dx/d\vartheta=dy/d\vartheta=d^2x/d\vartheta^2=d^2y/d\vartheta^2=0$, but if $\psi_{\vartheta\vartheta\vartheta}\neq 0$, then not both third derivatives vanish.

Theorem 12. *If a supersonic limit line enters the point, then its image is either perpendicular or tangent to the sonic circle.*

Proof. Since the point lies on a sonic limit line, the first nonvanishing derivative must be of the type $\psi_{\underbrace{\vartheta\ldots\vartheta}_{1+n}}\neq 0$ (see Section 2). Then the expansion of Theorem 2 holds:

$$F=-\frac{\gamma+1}{q_t^3}\,\overline{k}^2(\delta q)\,(\delta\vartheta)^{2n}+O(\delta q,\delta\vartheta)^{2n+2},$$

and the curve on which $F(q,\vartheta)=0$ is either tangent $(\delta q=0)$ or perpendicular $(\delta\vartheta=0)$ to the sonic circle.

Theorem 13. *If $\psi_{\vartheta\vartheta\vartheta}\neq 0$, then the streamline $\psi(q,\vartheta)=c$ through the image point has a triple point with three branches entering from the subsonic side, and one from the supersonic side, all in the radial direction. Two of the branches on the subsonic side form a cusp of the first kind.*

Proof. As in Theorem 11, let $\bar\psi=\psi-c$. Then

$$\bar\psi=\bar\psi_q=\bar\psi_\vartheta=\bar\psi_{qq}=\bar\psi_{q\vartheta}=\bar\psi_{\vartheta\vartheta}=\bar\psi_{qqq}=\bar\psi_{qq\vartheta}=\bar\psi_{q\vartheta\vartheta}=0,$$

$$\bar\psi_{\vartheta\vartheta\vartheta}\neq 0,\qquad \bar\psi_{qqqq}=0,\qquad \bar\psi_{qqq\vartheta}=-\frac{\gamma+1}{q_t^3}\,\bar\psi_{\vartheta\vartheta\vartheta}\neq 0,\qquad \bar\psi_{qqqqq}=0,$$

and the Taylor expansion about the point is

$$\bar\psi(q,\vartheta)=\frac{1}{3!}\left[\bar\psi_{\vartheta\vartheta\vartheta}(\delta\vartheta)^3\right]+\frac{1}{4!}\left[\tbinom{4}{1}\bar\psi_{qqq\vartheta}(\delta q)^3(\delta\vartheta)\right]+\cdots+$$
$$+\frac{1}{5!}\left[\tbinom{5}{1}\bar\psi_{qqqq\vartheta}(\delta q)^4(\delta\vartheta)+\cdots\right]+\cdots.$$

Thus any branch of the curve $\bar\psi(q,\vartheta)=0$ coming into the point has the direction $\delta\vartheta=0$. Now let $b=3!\,\bar\psi_{qqq\vartheta}/\bar\psi_{\vartheta\vartheta\vartheta}=-3!\,(\gamma+1)/q_t^3<0$ and $b'=3\,\bar\psi_{qqqq\vartheta}/\bar\psi_{\vartheta\vartheta\vartheta}$ and put $\delta\vartheta=u\,x$, $\delta q=x$. Then

$$G(x,u)=\frac{3!\,\bar\psi(q,\vartheta)}{\bar\psi_{\vartheta\vartheta\vartheta}}\cdot\frac{1}{x^3}=u^3+x\left[b\,u+\cdots\right]+x^2\left[b'\,u+\cdots\right]+\cdots=0,$$

and the curve $G(x,u)=0$ enters along $b\,u\,x=0$, *i.e.* $x=0$ or $u=0$.

By repeating the argument, we find as in the proof of Theorem 9 that $x=0$ corresponds to a unique curve, whose image in the hodograph

$$(\delta\vartheta)^2=-\,b\,(\delta q)^3+\cdots$$

forms a cusp of the first kind. Since $b<0$, this falls on the subsonic side of the sonic circle.

By a similar analysis, $u=0$ may be shown to correspond to a unique curve, whose image in the hodograph passes through the point in the radial direction.

5. Sonic point of a supersonic limit line: Ordinary point $(\psi_\vartheta\neq 0)$

(Examples of such a point are found in the flows of RINGLEB and of TEMPLE & YARWOOD. See *e.g.* [5].)

Theorem 14. *The supersonic limit line is unique and is tangent to the sonic circle at the image point.*

Proof. As in Theorem 1, we have $F=F_\vartheta=0$, $F_q\neq 0$. Hence there is a unique line $q=f(\vartheta)$ through the point on which $F(q,\vartheta)=0$, and it has the direction

$$\frac{dq}{d\vartheta}=-\frac{F_\vartheta}{F_q}=0,\qquad q=\text{const}.$$

Theorem 15. *The geometry in both physical and hodograph planes is the same as for an ordinary point of a sonic limit line.*

Proof. The directions of the isovel, equipotential, characteristics, streamline and isocline in the hodograph plane depend only on the property $\psi_q = 0$, $\psi_\vartheta \neq 0$ at the point. Moreover the effect on these curves of the mapping from the hodograph into the physical plane depends only on $\psi_q = 0$, $\psi_\vartheta \neq 0$ at the point. (*Cf.* Section 3 and [2].)

6. Sonic point of a supersonic limit line: Higher-order point ($\psi_\vartheta = 0$)

(i) *First non-vanishing derivative is of the type* $\psi_{\underbrace{\vartheta \ldots \vartheta}_{1+n}} \neq 0$. (The simplest example of this type of flow is one where $n = 1$, *i.e.* at the point: $\psi_q = \psi_\vartheta = \psi_{qq} = \psi_{q\vartheta} = 0$, $\psi_{\vartheta\vartheta} \neq 0$. Such a flow can be constructed by superposition of Chaplygin solutions as follows:

$$\psi \doteq A\vartheta + B \int \frac{\varrho}{q}\, dq + \frac{1}{q} (\sin\vartheta + \cos\vartheta) + C F_2(q) \sin 2\vartheta$$

where $F_2(q) = q^2 F(\tfrac{5}{3}, -3, 3, q^2)$, $A = -[q_t F_2'(q_t) + F_2(q_t)]/q_t^2 F_2'(q_t)$, $B = 1/\varrho_t q_t$, and $C = 1/2q_t^2 F_2'(q_t)$.)

Theorem 16. *The images of the branches of the supersonic limit line come in tangentially or perpendicularly to the sonic circle.*

Proof. As in Theorem 2 the Taylor expansion of F about the point is

$$F = -\frac{\gamma+1}{q_t^3} \bar{k}^2 (\delta q) (\delta\vartheta)^{2n} + O(\delta q, \delta\vartheta)^{2n+2};$$

hence the curve on which $F \equiv 0$ has the direction $\delta q = 0$ and/or $\delta\vartheta = 0$.

(ii) *First non-vanishing derivative (q decreasing) is of the type* $\psi_{q\underbrace{\vartheta \ldots \vartheta}_{1+n}} \neq 0$. (The simplest example of this type of flow is one where $n = 1$, *i.e.* $\psi_q = \psi_\vartheta = \psi_{qq} = 0$, $\psi_{q\vartheta} \neq 0$. Such a point is $q = q_t$, $\vartheta = 0$ in the flow $\psi = k\vartheta + (1/q) \sin\vartheta$, where $k = -1/q_t$.)

Theorem 17. *The images of the branches of the supersonic limit line come in perpendicularly to the sonic circle.*

Proof. Let $\frac{1}{n!} \psi_{q\underbrace{\vartheta \ldots \vartheta}_{1+n}} = k' \neq 0$. Then as in Theorem 2 we find the Taylor series

$$F = k'^2 (\delta\vartheta)^{2n} + O(\delta q, \delta\vartheta)^{2n+1}.$$

Hence $F \equiv 0$ has the direction $\delta\vartheta = 0$.

This research was supported in part by the United States Air Force under Contract No. AF 49(638)-154, monitored by the AF Office of Scientific Research of the Air Research and Development Command, and in part by the Office of Ordnance Research, U. S. Army, under Contract DA-36-034-ORD-1486 with the University of Maryland.

Ludford was a fellow of the John Simon Guggenheim Memorial Foundation in 1957—1958.

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(Received April 5, 1958)

Sonic Limit Singularities. Part II: Examples

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Communicated by Hilda Geiringer

1. Introduction

It was shown by Hilda Geiringer [*1*] that in addition to the well-known supersonic limit lines in the two-dimensional inviscid potential flow of a compressible fluid there may occur limit lines (exhibiting a different geometry) which coincide entirely or in part with the sonic line. This corrected an earlier statement by Ringleb [*2*], that the only flow exhibiting such a sonic limit line is the well-known radial or source flow.

In the first part of the present paper we gave the general theory of sonic limit singularities. Here we discuss special types of sonic limit line flow, providing tangible examples of this theory. In particular, three such flows are explicitly calculated, the resulting flow patterns being of some interest in themselves.

In a previous paper [*3*] the present authors showed how flows containing sonic limit lines may be constructed by forming suitable product solutions [see (1) in next section] of the linear hodograph equations, for any value of the separation constant $n \geqq 0$. As an example of a sonic limit line flow with no source-like character, we explicitly constructed the flow corresponding to $n=1$. (The case $n=0$ leads to the source flow.)

In this paper we investigate the case of general n (which may be supposed $\geqq 0$) and show that the corresponding class of flows has certain well-defined geometrical features. For n less than a certain value n_0 the flow in the physical plane represents a compressible analogue of the incompressible flow about a corner* (with a certain corner angle) and must be envisaged as taking place on a quadruply-sheeted surface. The sheets are joined at a supersonic limit line and at the sonic limit line which is a portion of a hypocycloid ($n>1$) or epicycloid ($n<1$). In certain cases, portions of the same sheet overlap in the large.

In addition we discuss the shape of the sonic limit line when solutions corresponding to different n are superposed and show how then the supersonic limit line can be eliminated so that an isolated sonic limit line is obtained.

Throughout the discussion we will make use of the general results established in Part I (this journal, Vol. 2, pp. 160—172), which will be referred to by means of the prefix I.

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* For a discussion of the corner flows corresponding to Chaplygin's product solutions, see [*4*] and references cited there.

2. Sonic limit line flows for any n

It was shown in [3] that flows with a sonic limit line may be constructed for any positive real number n, by taking the stream-function to be*

$$\psi(\tau,\vartheta) = P_n(\tau)\sin n\vartheta \equiv \tau^{\frac{1}{2}n} H_n(\tau)\sin n\vartheta, \qquad \tau = \frac{q^2}{q_m^2}, \tag{1}$$

where $H_n(\tau)$ is, in general, a combination of hypergeometric functions,

$$H_n(\tau) = \varkappa_n F_n^{(1)}(\tau) + \tau^{-n} F_n^{(2)}(\tau), \tag{2}$$

satisfying the equation

$$\tau(1-\tau)H_n'' + [(n-1) - (n+1-\beta)\tau]H_n' + \tfrac{1}{2}n(n+1)\beta H_n = 0, \tag{3}$$

the constant $\varkappa_n$ in the combination being chosen such that

$$\frac{dP_n}{d\tau} = \frac{d}{d\tau}\left[\tau^{\frac{1}{2}n} H_n(\tau)\right] = 0 \quad \text{at the sonic value} \quad \tau = \tau_t = \frac{1}{2\beta+1} \quad \left(\beta = \frac{1}{\gamma-1}\right). \tag{4}$$

Any linear combination of such solutions (1) will also be a solution containing a sonic limit line. The discussion of these composite flows will be deferred until Section 6.

Corresponding to this solution in the τ,ϑ-plane (hodograph), the coordinate functions, see (I.2), are given parametrically in terms of τ and ϑ by

$$\begin{aligned} x_n &= \frac{1}{2}\tau^{-\frac{1}{2}}(1-\tau)^{-\frac{5}{2}}\Big[-\frac{1}{n+1}\Big(2\tau\frac{dP_n}{d\tau} - nP_n\Big)\cos(n+1)\vartheta - \\ &\qquad\qquad - \frac{1}{n-1}\Big(2\tau\frac{dP_n}{d\tau} + nP_n\Big)\cos(n-1)\vartheta\Big] \\ y_n &= \frac{1}{2}\tau^{-\frac{1}{2}}(1-\tau)^{-\frac{5}{2}}\Big[-\frac{1}{n+1}\Big(2\tau\frac{dP_n}{d\tau} - nP_n\Big)\sin(n+1)\vartheta + \\ &\qquad\qquad + \frac{1}{n-1}\Big(2\tau\frac{dP_n}{d\tau} + nP_n\Big)\sin(n-1)\vartheta\Big] \end{aligned} \tag{5}$$

if $n \neq 1$. For $n = 1$, the coordinate functions were given in [3] and will not be repeated here. The case $n = 0$, leading to the well-known radial or source flow will also be omitted in the discussion.

We now investigate the shape of the sonic limit line in the physical plane corresponding to any $n > 0$. From (4) and (5) it is given by**

$$\begin{aligned} x_n &= \frac{nP_n(\tau_t)}{2\tau_t^{\frac{1}{2}}(1-\tau_t)^{\frac{5}{2}}}\left[\frac{1}{n+1}\cos(n+1)\vartheta - \frac{1}{n-1}\cos(n-1)\vartheta\right], \\ y_n &= \frac{nP_n(\tau_t)}{2\tau_t^{\frac{1}{2}}(1-\tau_t)^{\frac{5}{2}}}\left[\frac{1}{n+1}\sin(n+1)\vartheta + \frac{1}{n-1}\sin(n-1)\vartheta\right]. \end{aligned} \tag{6}$$

These are the parametric equations of curves (roulettes) which can be generated by rolling a circle on the outside or inside of a fixed circle. For $n > 1$ the ratio of the radius of the rolling circle to that of the fixed circle is $(n-1)/2n$ and the curves are *hypocycloids*. For $n<1$ this ratio is $(1-n)/2n$ and the curves are *epicycloids*. For $n=1$ (the case discussed in [3]) the curve is a *cycloid* and for $n=0$ (the radial flow) the roulette degenerates into a *circle*. When $2n/(n-1)$ is rational, the sonic limit line will close up after a certain number of revolutions. Thus for $n=\frac{1}{3}, \frac{1}{2}, 2, 3$ the curves are particularly simple, namely a cardioid, nephroid, astroid, and tricuspid, respectively.

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* In general $\psi(\tau,\vartheta) = P_n(\tau)\sin(n\vartheta + \varepsilon_n) = \tau^{\frac{1}{2}n} H_n(\tau)\sin(n\vartheta + \varepsilon_n)$. Since the inclusion of the phase angle ε_n merely amounts to a rotation of axes in the physical plane, there is no loss in generality in taking ε_n to be zero throughout the following discussion.

** The same result follows from Theorem I.7.

In Part I we called a point where $\psi_\tau=\psi_\vartheta=0$ for $\tau=\tau_t$ a higher-order sonic limit point. Thus for any n, the sonic limit line will have higher-order points where

$$\psi_\vartheta = \tau_t^{\frac{1}{2}n}\, H_n(\tau_t)\, n \cos n\vartheta = 0,$$

that is, at the points corresponding to

$$\vartheta = \frac{2m+1}{n}\cdot\frac{1}{2}\pi \qquad m=0,\ \pm 1,\ \pm 2, \ldots. \tag{7}$$

It is precisely at these points that the roulettes have their cusps. A sonic limit line does not necessarily have cusps at its higher-order points, however. This depends on the vanishing of $\psi_{\vartheta\vartheta}$ (*cf.* Section I.4).)

3. The associated supersonic limit line and streamlines in the hodograph

In addition to sonic limit lines the flows given by (1) also possess other (supersonic) limit singularities (*cf.* Theorem I.8). The Jacobian of the transformation from the hodograph to the physical plane shows that these occur where

$$\frac{\partial(x,y)}{\partial(\tau,\vartheta)} = -\frac{2\tau}{(1-\tau)^\beta}\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{(2\beta+1)\,\tau-1}{2\tau(1-\tau)^{\beta+1}}\left(\frac{\partial\psi}{\partial\vartheta}\right)^2 = 0. \tag{8}$$

For the solutions (1) these supersonic limit lines are the curves

$$\cot n\vartheta = \mp\sqrt{\frac{1-\tau}{(2\beta+1)\,\tau-1}}\cdot\frac{2\tau}{n}\cdot\frac{1}{P_n}\cdot\frac{dP_n}{d\tau}. \tag{9}$$

To investigate the shape of these curves we note from (3) that the function $P_n(\tau)$ satisfies the equation

$$\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\,\frac{dP_n}{d\tau}\right] - \frac{n^2}{4}\cdot\frac{1-(2\beta+1)\,\tau}{\tau(1-\tau)^{\beta+1}}\,P_n = 0. \tag{10}$$

Hence if we let

$$S_n = \frac{2\tau}{n}\cdot\frac{1}{P_n}\cdot\frac{dP_n}{d\tau}, \tag{11}$$

then S_n will satisfy the Riccati equation

$$\frac{dS_n}{d\tau} + \frac{\beta}{1-\tau}\,S_n + \frac{n}{2\tau}\left[S_n^2 - \frac{1-(2\beta+1)\,\tau}{1-\tau}\right] = 0. \tag{12}$$

At the sonic circle $\tau=\tau_t=\dfrac{1}{2\beta+1}$ we have the conditions

$$S_n = 0, \qquad \frac{dS_n}{d\tau} = 0 \tag{13}$$

which follow from (4) and (12).

Now the expression on the right of (9) becomes indeterminate at $\tau = \tau_t$, but application of L'Hospital's rule leads to

$$(14) \qquad \cot n\vartheta = 0, \quad \vartheta = \frac{2m+1}{n} \cdot \frac{1}{2}\pi \qquad m = 0, \pm 1, \pm 2, \ldots.$$

Hence the supersonic limit line intersects the sonic limit line at the higher-order points (7). This was seen to be a general property of sonic limit lines in Section I.4. Moreover it was shown in Theorem I.9 that, when $\psi_{\vartheta\vartheta} \neq 0$, the supersonic limit line enters the higher-order point in a cusp. At the maximum circle $\tau = 1$, again (14) holds. For $(2m/n) \cdot \frac{1}{2}\pi < \vartheta < [(2m+2)/n] \cdot \frac{1}{2}\pi$ the supersonic limit line must extend from $\tau = \tau_t$ to $\tau = 1$, in two symmetric branches (corresponding to the plus and minus signs in (9)) about the line of symmetry $\vartheta = [(2m+1)/n] \cdot \frac{1}{2}\pi$.

In order to determine the number of sheets required to represent the flow in the physical plane it is important to know whether the branches of the supersonic limit line in the hodograph plane intersect again on the line $\vartheta = [(2m+1)/n] \cdot \frac{1}{2}\pi$ in the interval $\tau_t < \tau < 1$. Such supersonic double limit points certainly occur for some combinations of $F^{(1)}(\tau)$ and $F^{(2)}(\tau)$ (see for example Craggs' doublet [5]). We shall show, however, that for any n less than a certain value n_0, which we do not compute explicitly, the flow represented by (2) does not possess such points.

From (9) and (11) such a double limit point would have to occur at a zero of S_n. Now at the initial point τ_t of the interval $\tau_t \leqq \tau \leqq 1$ we have, on differentiating (12) and using (13), that

$$\frac{d^2 S_n}{d\tau^2} = -\frac{n(2\beta+1)}{2\tau(1-\tau)^2} < 0,$$

so that from (13), $S_n(\tau)$ is negative in the neighborhood of τ_t. Thus if the curve $S_n = S_n(\tau)$ is continuous in the interval $\tau_t < \tau < 1$, it cannot cross or touch the τ-axis in this interval; for it would have to do so with non-negative slope, whereas from (12) we see that at a zero of S_n in the supersonic region $dS_n/d\tau$ would have to be negative.

We now investigate the possibility that $S_n = S_n(\tau)$ has an infinity in the interval $\tau_t < \tau < 1$. Consider the zero isoclines of (12). The ordinates of these in the τ, S_n-plane are given by

$$S_n = \frac{-\beta\tau \pm \sqrt{\beta^2\tau^2 - n^2(1-\tau)\,[(2\beta+1)\tau - 1]}}{n(1-\tau)}$$

and are real when

$$\Delta = \beta^2 - n^2\left(\frac{1}{\tau} - 1\right)\left(2\beta + 1 - \frac{1}{\tau}\right) \geqq 0.$$

Now the maximum of $(1/\tau - 1)(2\beta + 1 - 1/\tau)$ occurs at $\tau = 1/\beta + 1$ and is β^2. Hence for any $n \leqq 1$, we have $\Delta \geqq 0$ in the whole interval, and the zero isoclines extend from $\tau = 0$ to $\tau = 1$ with non-positive slope. From the sketch (Fig. 1) it is then clear that the integral curve $S_n(\tau)$ which satisfies the conditions (13) at $\tau = \tau_t$ cannot have an infinity in the range $\tau_t < \tau < 1$, for it must lie below the τ-axis and cannot cross the zero isoclines, which therefore bound it from below. Thus $S_n < 0$ for all $\tau_t < \tau < 1$.

Consider now the case $n>1$. For $n>1$, we have $\Delta<0$ for some values of τ, and the zero isoclines lie as shown in Fig. 2. Thus the integral curve $S_n(\tau)$ which satisfies (13) is no longer bounded from below by the isoclines, and may have a singularity at some point τ_0 in the interval $\tau_t<\tau<1$ such that $\lim\limits_{\tau\to\tau_0} S_n(\tau)=-\infty$ and $\lim\limits_{\tau\to\tau_0^+} S_n(\tau)=+\infty$. As shown in the figure, the branch on which $\lim\limits_{\tau\to\tau_0^+} S_n(\tau)=+\infty$ will then cross the τ-axis with negative slope, as required by (12).

It can be shown from the asymptotic behavior of (12) that such infinities do in fact occur for large enough values of n; moreover, when there is an infinity (and hence a zero) for one value of n there is also for all larger values. The smallest value of n for which the integral curve has such a singularity will be denoted by n_0 (just proved greater than unity). This value n_0 for which $S_n(\tau)$ first has zeroes will not be calculated explicitly. However, the example worked out in Section 5 for the case $n=2$ exhibits no supersonic double limit points, so that we may conclude $n_0>2$.

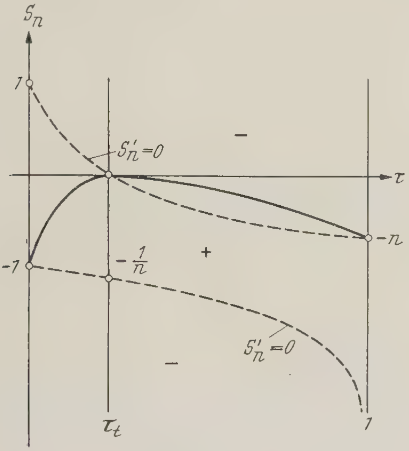

Fig. 1*. $n<1$

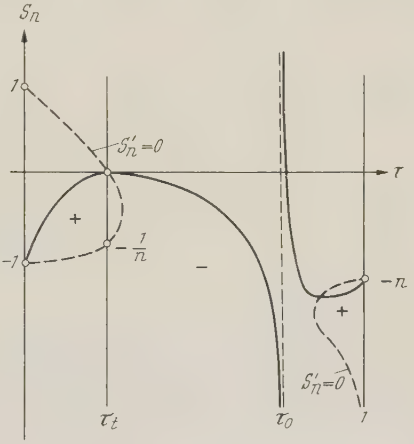

Fig. 2*. $1<n_0<n$

From these results it is clear then, that for any $n<n_0$ the supersonic limit line in the hodograph plane in a flow given by (1) consists, for $(2m/n)\cdot\frac{1}{2}\pi<\vartheta<[(2m+2)/n]\cdot\frac{1}{2}\pi$, of a drop-shaped portion with the line $\vartheta=[(2m+1)/n]\cdot\frac{1}{2}\pi$ as axis of symmetry (see for example Fig. 7). We will not discuss the shape of the supersonic limit line when $n\geqq n_0$.

We now turn to a discussion of the streamlines in the hodograph plane for the case $n<n_0$. The straight streamlines are given by $d\psi=\psi_\tau\,d\tau+\psi_\vartheta\,d\vartheta=0$, $d\vartheta=0$ which implies $\psi_\tau=0$, or from (1):

$$\vartheta=m\cdot\frac{\pi}{n},\qquad m=0,\ \pm1,\ \pm2,\ldots.$$

By symmetry we need only consider the circular sector $0<\vartheta<\pi/n$. If the straight streamlines are interpreted as boundaries in the physical plane it is evident that (1) represents flow about a corner of angle $\pi-(\pi/n)$. The total angle through which the stream turns is π/n.

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* The plus and minus signs indicate regions in which the integral curves have positive and negative slope, respectively.

Let us now consider the non-straight streamlines in the hodograph. Differentiating (1) along a streamline

$$d\psi = \frac{dP_n}{d\tau} \sin n\vartheta \, d\tau + n P_n \cos n\vartheta \, d\vartheta = 0,$$

so that the inclination α of the streamline to the radius vector at any point of the hodograph is given by

$$\cot \alpha = \frac{1}{2\tau} \cdot \frac{d\tau}{d\vartheta} = -\frac{\cot n\vartheta}{2\tau} \cdot \frac{P_n}{P_n'}$$

and on substituting (1) we find

$$\cot \alpha = -\frac{\cot n\vartheta}{S_n}. \tag{15}$$

Now we have seen above that

$$S_n = 0 \quad \text{at} \quad \tau = \tau_t; \qquad S_n < 0 \quad \text{throughout} \quad \tau_t < \tau < 1.$$

By a similar argument to that in the proof above, it can be shown that

$$S_n < 0 \quad \text{throughout} \quad 0 < \tau < \tau_t$$

so that for $n < n_0$,

$$\operatorname{sign}(\cot \alpha) = \operatorname{sign}(\cot n\vartheta) \quad \text{throughout} \quad 0 < \tau < 1.$$

Since all streamlines pass through the origin, it is clear then, that as ϑ increase from 0 to π/n the streamlines form closed loops, which are symmetric about the line $\vartheta = \pi/2n$ (see *e.g.* Fig. 7). The curvature of a streamline changes continuously, and becomes zero at the sonic circle $\tau = \tau_t$ (in accord with the theory of the sonic limit line, see [3] and Theorem I.6). The only exception to this general behavior of the non-straight streamlines is the streamline, say ψ_c, which passes through the higher-order sonic limit point at $\tau = \tau_t$, $\vartheta = \pi/2n$. This is given by

$$\psi_c = P_n(\tau_t) \sin n\left(\frac{\pi}{2n}\right) = P_n(\tau_t)$$

and is known to have a cusp at that point (Theorem I.11). It divides the streamlines which do not intersect a limit line — sonic, l_t, or supersonic, l — from those that do, and will be called the *critical streamline* ψ_c. Another streamline of interest is the one passing through the point $\tau = 1$, $\vartheta = \pi/2n$. Given by

$$\psi_a = P_n(1) \sin n\left(\frac{\pi}{2n}\right) = P_n(1),$$

it will be called the *asymptotic streamline*, for reasons which will become clear in the next section.

The streamlines in the hodograph plane may then be divided into three classes:

(a) those which intersect neither the l_t nor the l: $\psi_c < \psi < \infty$,
(b) those which intersect both the l_t and the l: $\psi_a < \psi < \psi_c$,
(c) those which intersect only the l_t: $0 < \psi < \psi_a$.

The behaviors of these three types of streamlines in the physical plane are radically different.

4. The physical plane

Consider now the flow in the physical plane corresponding to the hodograph solution in the typical circular sector $0<\tau<1$, $0<\vartheta<\pi/n$. From the geometry in the neighborhood of limit lines (sonic and supersonic) we know that the streamlines are cusped there and consequently the regions to one side are multiply covered. Thus the limit lines in the physical plane delineate regions in which the flow changes its multi-valued character. This can be envisaged as a folding of the physical plane into a multiply-sheeted configuration with folds at the limit lines, so that each sheet is simply covered by the appropriate portion of the total flow, and at the limit line folds the flow passes from one sheet to the appropriate next sheet. Due to our choice of the representative circular sector in the hodograph the entire multi-sheeted configuration representing the physical plane is of course bounded by the map of the sides of the sector, *i.e.* the straight streamlines $\vartheta=0$, $\vartheta=\pi/n$.

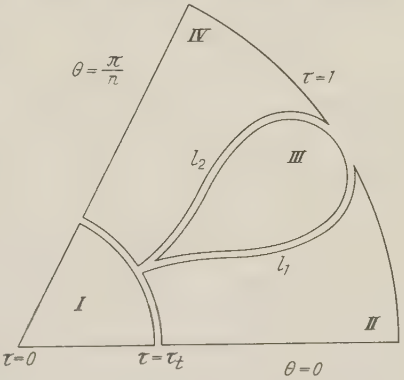

Fig. 3. Circular sector in the hodograph plane

Accordingly, for $n<n_0$, the physical plane must be envisaged as a quadruply-sheeted surface, the sheets corresponding to the regions in the hodograph bounded by (see Fig. 3):

$$\text{(I)}\qquad 0<\tau<\tau_t,\ \vartheta=0;\quad \tau=\tau_t,\ 0<\vartheta<\frac{\pi}{n};\quad 0<\tau<\tau_t,\ \vartheta=\frac{\pi}{2n};$$

$$\text{(II)}\qquad \tau_t<\tau<1,\ \vartheta=0;\quad \tau=1,\ 0<\vartheta<\frac{\pi}{2n};\quad l_1;\ \tau=\tau_t,\ 0<\vartheta<\frac{\pi}{2n};$$

$$\text{(III)}\qquad l_1;\quad l_2;$$

$$\text{(IV)}\qquad l_2;\quad \tau=1,\ \frac{\pi}{2n}<\vartheta<\frac{\pi}{n};\quad \tau_t<\tau<1,\ \vartheta=\frac{\pi}{n};\quad \tau=\tau_t,\ \frac{\pi}{2n}<\vartheta<\frac{\pi}{n}.$$

Examples (Section 5, Figs. 6, 9) show that portions of sheet I may overlap.

We have already discussed the shape of the sonic limit line in the physical plane. The supersonic limit line maps into a wedge-shaped curve with a cusp at the higher-order sonic limit point, the cusp direction coinciding with that of the sonic limit line.

The streamlines in the physical plane behave as follows*. Initially all streamlines have the $\vartheta=0$ direction. They then bend around in a clockwise direction and finally all assume the $\vartheta=\pi/n$ direction. Between these two final states their behavior depends on the group to which they belong in our classification of streamlines in the hodograph plane (see end of Section 3):

(a) These streamlines bend smoothly about the cusp of the $\mathscr{L}_t$ and never leave sheet I.

---

* Examples are given by Figs. 4, 5 and 7, 8.

(b) These start on sheet I, are cusped at the $\mathscr{L}_t$ and pass onto sheet II, continue on sheet II until they meet the $\mathscr{L}_1$, reflect onto sheet III, continue on sheet III until they reach the $\mathscr{L}_2$, pass there to sheet IV, continue on sheet IV up to the $\mathscr{L}_t$, and reflect back onto sheet I.

(c) These start on sheet I, reflect at one branch of the $\mathscr{L}_t$ onto sheet II and continue on this sheet to infinity (the circle $\tau = 1$ maps into infinity in the physical plane); they then return on sheet IV, and reflect at the other branch of the $\mathscr{L}_t$ back onto sheet I.

Note that the dividing streamline between group (b) and group (c), namely ψ_a, asymptotically approaches the two branches of the supersonic limit line.

An example ($n = 1$) having the features described above was worked out in [*3*]. Other such flows will be constructed explicitly in the next section.

5. The sonic limit line flows for $n = 2$ and $n = \frac{1}{2}$

As a representative example of the above results consider $n = 2$, $\gamma = \frac{7}{5}$. The straight streamlines are $\vartheta = 0$ and $\vartheta = \frac{1}{2}\pi$, corresponding to a compressible flow about a 90° corner. This particular choice of n is exceptional, in that one of the solutions of the hypergeometric equation (3) is a polynomial, and the other is an infinite series involving a logarithmic term.

The streamfunction in this case is taken to be

$$\psi = P_2(\tau) \sin 2\vartheta = \left[\varkappa_2 \tau F^{(1)}(\tfrac{5}{2}, -3, 3; \tau) + \tau^{-1} F^{(2)}(\tfrac{1}{2}, -5, -1; \tau)\right] \sin 2\vartheta, \tag{16}$$

where

$$F^{(1)}\left(\frac{5}{2}, -3, 3; \tau\right) = \frac{1}{32} (32 - 80\tau + 70\tau^2 - 21\tau^3),$$

$$F^{(2)}\left(\frac{1}{2}, -5, -1; \tau\right)$$
$$= 1 + \frac{5}{2}\tau - \frac{15}{2}\tau^2 F\left(\frac{5}{2}, -3, 3; \tau\right) \log\tau + \frac{120}{\Gamma(\frac{1}{2})} \sum_{\nu=3}^{\infty} \frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(\nu-5)}{\Gamma(\nu-1)\,\nu!} \tau^\nu,$$

$$\varkappa_2 = -\left[\frac{d}{d\tau}(\tau^{-1}F^{(2)}) \Big/ \frac{d}{d\tau}(\tau F^{(1)})\right]_{\tau=\frac{1}{6}} = 123.12.$$

The coordinate functions are, from (5),

$$\begin{aligned} x_2 &= \tau^{-\frac{1}{2}}(1-\tau)^{-\frac{5}{2}}\left[\tfrac{1}{3}(\tau P_2' - P_2)\cos 3\vartheta + (\tau P_2' + P_2)\cos\vartheta\right] \\ y_2 &= \tau^{-\frac{1}{2}}(1-\tau)^{-\frac{5}{2}}\left[\tfrac{1}{3}(\tau P_2' - P_2)\sin 3\vartheta - (\tau P_2' + P_2)\sin\vartheta\right] \end{aligned} \tag{17}$$

and the sonic limit line in the physical plane is consequently the astroid

$$x_2 = -\tfrac{1}{3}\left[\tau_t^{-\frac{1}{2}}(1-\tau_t)^{-\frac{5}{2}} P_2(\tau_t)\right](\cos 3\vartheta - 3\cos\vartheta),$$
$$y_2 = -\tfrac{1}{3}\left[\tau_t^{-\frac{1}{2}}(1-\tau_t)^{-\frac{5}{2}} P_2(\tau_t)\right](\sin 3\vartheta + 3\sin\vartheta),$$

or, on eliminating ϑ,

$$(x_2 + y_2)^{\frac{2}{3}} + (x_2 - y_2)^{\frac{2}{3}} = \left[4\sqrt{2}\,\tau_t^{-\frac{1}{2}}(1-\tau_t)^{-\frac{5}{2}} P_2(\tau_t)\right]^{\frac{2}{3}}.$$

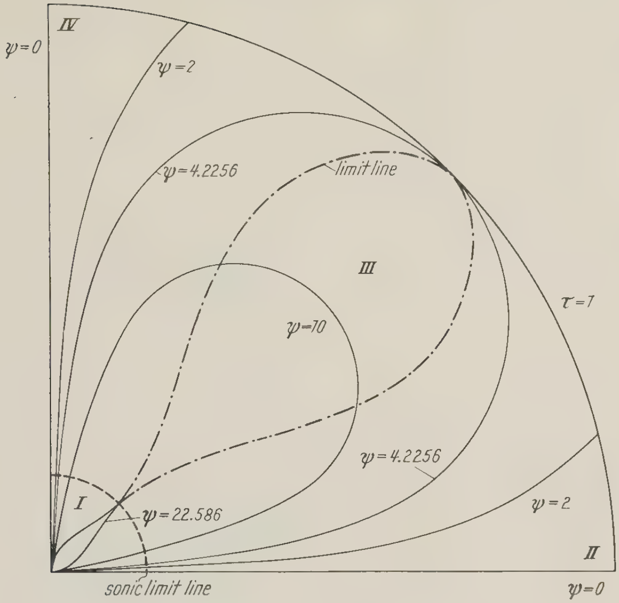

Fig. 4. Case $n = 2$. Hodograph plane

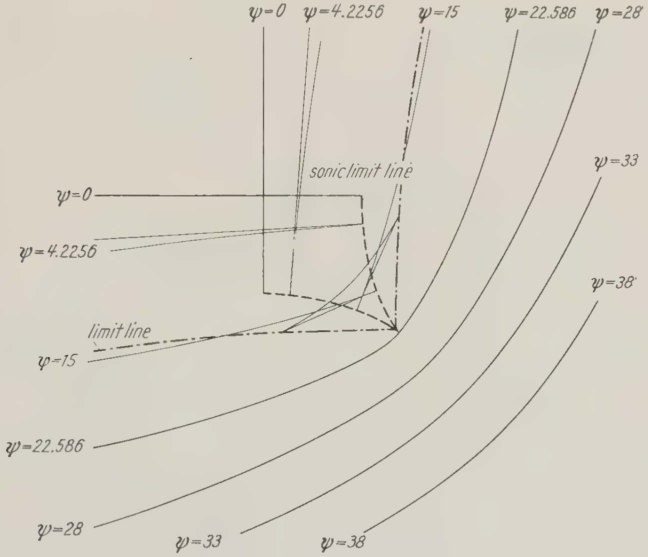

Fig. 5. Case $n = 2$. Physical plane

The supersonic limit line in the hodograph is given by

$$\tan 2\vartheta = \pm \frac{1}{\tau}\sqrt{\frac{6\tau - 1}{1 - \tau}} \cdot \frac{P_2}{P_2'}\,, \tag{18}$$

and its image in the physical plane is obtained from (17).

The critical streamline is

$$\psi_c = P_2(\tau_t) = 22.586,$$

and the asymptotic streamline is*

$$\psi_a = P_2(1) = \varkappa_2 F^{(1)}\left(\frac{5}{2}, -3, 3; 1\right) + F^{(2)}\left(\frac{1}{2}, -5, -1; 1\right)$$
$$- \varkappa_2 \frac{\Gamma(3)\,\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})\,\Gamma(6)} + \frac{\Gamma(-1)\,\Gamma(\frac{7}{2})}{\Gamma(-\frac{3}{2})\,\Gamma(4)} = 4.2256.$$

The streamline corresponding to any $\psi = \psi_0$ is obtained from (16) and (17).

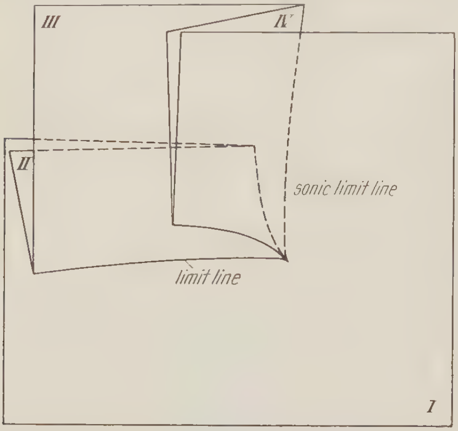

Fig. 6. Case $n = 2$. Sketch of multi-sheeted surface corresponding to physical plane

Graphs of the limit lines and selected streamlines in the hodograph and in the physical plane, respectively, are presented in Figs. 4 and 5**. The manner in which the sheets corresponding to the physical plane are connected is indicated diagrammatically in Fig. 6.

As another example of the foregoing, this time exhibiting a sonic limit line of the epicycloidal type (namely a nephroid), we select the case $n = \frac{1}{2}$. Here both solutions of the hypergeometric equation are unexceptional and are represented by infinite series:

$$(19)\quad \psi(\tau, \vartheta) = P_{\frac{1}{2}}(\tau) \sin \tfrac{1}{2}\vartheta = \left[\varkappa_{\frac{1}{2}} \tau^{\frac{1}{4}} F^{(1)}(a, b, \tfrac{3}{2}; \tau) + \tau^{-\frac{1}{4}} F^{(2)}(a - \tfrac{1}{2}, b - \tfrac{1}{2}, \tfrac{1}{2}; \tau)\right] \sin \tfrac{1}{2}\vartheta$$

where

$$a, b = \tfrac{1}{4}\left(-4 \pm \sqrt{31}\right),$$

$$F^{(1)}\left(a, b, \frac{3}{2}; \tau\right) = \frac{\Gamma(\frac{3}{2})}{\Gamma(a)\,\Gamma(b)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu + a)\,\Gamma(\nu + b)}{\Gamma(\nu + \frac{3}{2})\,\nu!} \tau^{\nu},$$

$$F^{(2)}\left(a - \frac{1}{2}, b - \frac{1}{2}, \frac{1}{2}; \tau\right) = \frac{\Gamma(\frac{1}{2})}{\Gamma(a - \frac{1}{2})\,\Gamma(b - \frac{1}{2})} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu + a - \frac{1}{2})\,\Gamma(\nu + b - \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})\,\nu!} \tau^{\nu},$$

$$\varkappa_{\frac{1}{2}} = -\left[\frac{d}{d\tau}\left(\tau^{-\frac{1}{4}} F^{(2)}\right) \Big/ \frac{d}{d\tau}\left(\tau^{\frac{1}{4}} F^{(1)}\right)\right]_{\tau = \frac{1}{6}} = 3.42054.$$

The mapping to the physical plane is given by

$$(20)\quad \begin{aligned} x_{\frac{1}{2}} &= \tau^{-\frac{1}{2}}(1 - \tau)^{-\frac{5}{2}}\left[-\tfrac{1}{3}\left(2\tau P'_{\frac{1}{2}} - \tfrac{1}{2} P_{\frac{1}{2}}\right) \cos \tfrac{3}{2}\vartheta + \left(2\tau P'_{\frac{1}{2}} + \tfrac{1}{2} P_{\frac{1}{2}}\right) \cos \tfrac{1}{2}\vartheta\right], \\ y_{\frac{1}{2}} &= \tau^{-\frac{1}{2}}(1 - \tau)^{-\frac{5}{2}}\left[-\tfrac{1}{3}\left(2\tau P'_{\frac{1}{2}} - \tfrac{1}{2} P_{\frac{1}{2}}\right) \sin \tfrac{3}{2}\vartheta + \left(2\tau P'_{\frac{1}{2}} + \tfrac{1}{2} P_{\frac{1}{2}}\right) \sin \tfrac{1}{2}\vartheta\right], \end{aligned}$$

---

* The expansion

$$F(a, b, c; \tau) = \frac{\Gamma(c)\,\Gamma(c - a - b)}{\Gamma(c - a)\,\Gamma(c - b)} F(a, b, 1 + a + b - c; 1 - \tau) +$$
$$+ \frac{\Gamma(a + b - c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)} (1 - \tau)^{c - a - b} F(c - a, c - b, 1 + c - a - b; 1 - \tau)$$

is used near $\tau = 1$.

** In the actual computation of these flows use was made of the tables in [6] and [7], taking due account of the errors in [7] which were pointed out in [8].

and the sonic limit line is the nephroid

$$x_{\frac{1}{2}} = \tfrac{1}{2}\,\tau_t^{-\frac{1}{2}}(1-\tau_t)^{-\frac{5}{2}}\,P_{\frac{1}{2}}(\tau_t)\,\left(\tfrac{1}{3}\cos\tfrac{3}{2}\vartheta + \cos\tfrac{1}{2}\vartheta\right)$$
$$y_{\frac{1}{2}} = \tfrac{1}{2}\,\tau_t^{-\frac{1}{2}}(1-\tau_t)^{-\frac{5}{2}}\,P_{\frac{1}{2}}(\tau_t)\,\left(\tfrac{1}{3}\sin\tfrac{3}{2}\vartheta + \sin\tfrac{1}{2}\vartheta\right).$$

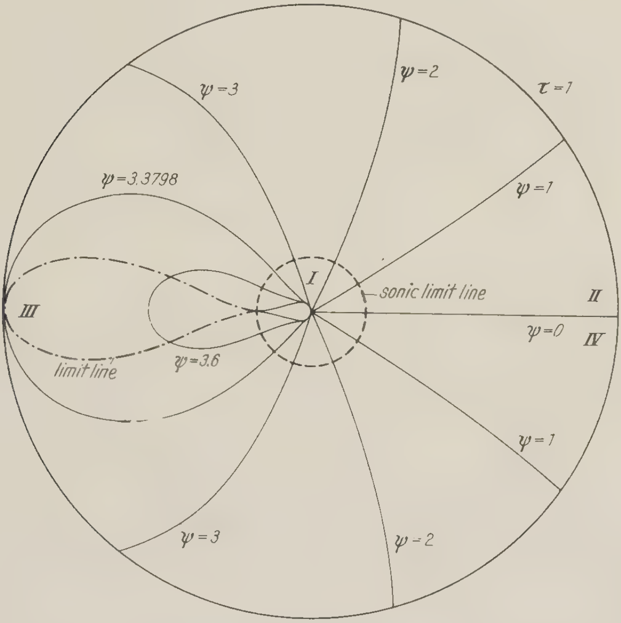

Fig. 7. Case $n = \frac{1}{2}$. Hodograph plane

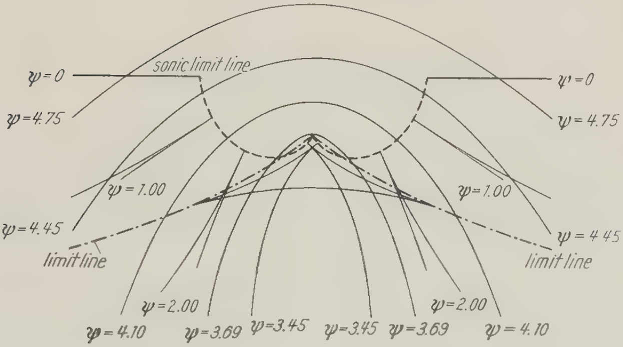

Fig. 8. Case $n = \frac{1}{2}$. Physical plane

The supersonic limit line in the physical plane is obtained from (20) and

$$\tan\frac{1}{2}\vartheta = \pm\frac{1}{4\tau}\sqrt{\frac{6\tau-1}{1-\tau}}\cdot\frac{P_{\frac{1}{2}}}{P'_{\frac{1}{2}}}. \tag{21}$$

The critical streamline is

$$\psi_c = P_{\frac{1}{2}}(\tau_t) = 3.685\,76,$$

and the asymptotic streamline is

$$\psi_a = P_{\frac{1}{2}}(1) = \varkappa_{\frac{1}{2}} F(a, b, \tfrac{3}{2}; 1) + F(a - \tfrac{1}{2}, b - \tfrac{1}{2}, \tfrac{1}{2}; 1)$$
$$= \varkappa_{\frac{1}{2}} \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2} - a - b)}{\Gamma(\frac{3}{2} - a)\,\Gamma(\frac{3}{2} - b)} + \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2} - a - b)}{\Gamma(1 - a)\,\Gamma(1 - b)} = 3.37976.$$

The streamlines in the physical plane are obtained from (19) and (20).

Graphs of the limit lines and selected streamlines in the hodograph and physical plane for this case are shown in Figs. 7 and 8. Fig. 9 is a diagrammatic representation of the sheeted surface corresponding to the physical plane.

An interesting feature of both these flows is that sheet I overlaps itself (*cf.* [2] and [3]).

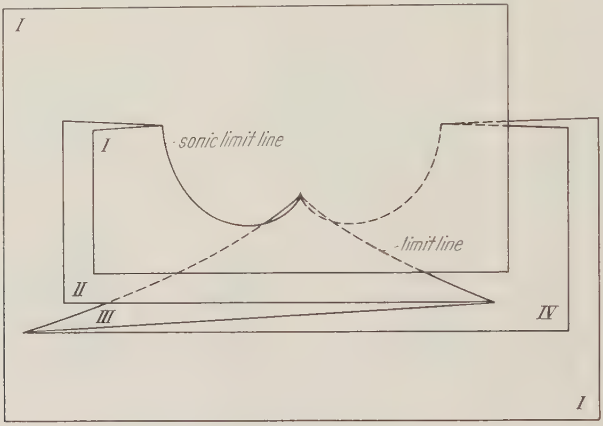

Fig. 9. Case $n = \frac{1}{2}$. Sketch of multi-sheeted surface corresponding to physical plane

6. Superposition of sonic limit line flows for different n. The isolated sonic limit line

Superposition of a finite number of sonic limit line solutions (1) for different n will again yield a flow with a sonic limit line. The coordinate functions may be found by adding the coordinate functions (5) of the individual flows. Naturally, the simple geometric character of the sonic limit line in the physical plane for the separate solutions is thereby destroyed.

However, a simple rule for the geometrical construction of this line may still be given. Draw the sonic limit lines which correspond to the individual solutions. Then add the coordinates of the points on the individual sonic limit lines at which the slopes of the latter are the same. The curve constructed in this manner will be the sonic limit line of the combined solution. This follows from the fact that a sonic limit line is always perpendicular to the streamline, so that one is in fact adding for the same value of ϑ, *i.e.* for the same point in the hodograph.

In Section 3 we noted that, for any n, the flow constructed from (1) necessarily contained a supersonic limit line in addition to the sonic limit line. The question arises whether, by superposition, a flow can be constructed which is devoid of any supersonic limit singularities and contains only a sonic limit line.

Now given the solution (1) for any n, we can, by adding to it a suitable radial flow solution ($n=0$), obtain a flow such that (i) there is still a sonic limit line and (ii) ψ_ϑ has the same sign along the sonic line. Hence the sonic line consists entirely of ordinary limit points ($\psi_\vartheta \neq 0$) and it must therefore be an isolated sonic limit line by Theorem I.5. In fact we shall see that even in the limiting case of the above construction, when ψ_ϑ is allowed to take zero values (without changing sign: $\psi_{\vartheta\vartheta}=0$, *cf.* Theorem I.8), the flow still has an isolated sonic limit line (see Section I.4 (ii); an example will be presented in Section 7).

For any n, we have for the solution (1)

$$\psi_\vartheta(\tau_t, \vartheta) = P_n(\tau_t)\, n \cos n\vartheta,$$

whereas for the radial flow, whose streamfunction is

$$\psi(\tau, \vartheta) = k\vartheta, \tag{22}$$

we have

$$\psi_\vartheta(\tau_t, \vartheta) = k.$$

If we add (1) and (22) and require that ψ_ϑ does not change sign anywhere on $\tau=\tau_t$, we must choose k such that

$$|k| \geqq \mp n\, P_n(\tau_t) \quad \text{according as} \quad P_n(\tau_t) \lessgtr 0. \tag{23}$$

Clearly the addition of the radial flow does not affect the vanishing of ψ_τ on the sonic line; hence the resulting flow will also possess a sonic limit line. Moreover if the strict inequality holds in (23) we have $\psi_\vartheta \neq 0$ for $\tau=\tau_t$ and the sonic limit line is isolated.

Consider now what happens to the supersonic limit line, which appears in the flow (1), when the radial solution (22), with $|k|$ increasing from zero, is added. From the streamfunction of the combined flow

$$\psi(\tau, \vartheta) = k\vartheta + P_n(\tau) \sin n\vartheta$$

we see that the supersonic limit line is given by

$$k + P_n\, n \cos n\vartheta \pm \sqrt{PQ}\, P_n' \sin n\vartheta = 0, \tag{24}$$

see (8), where

$$P = \frac{2\tau(1-\tau)^{\beta+1}}{(2\beta+1)\,\tau - 1}, \qquad Q = \frac{2\tau}{(1-\tau)^\beta}. \tag{25}$$

Equation (24) may be written

$$\cos\left[n\vartheta \mp \delta(\tau)\right] = \frac{k}{\sqrt{n^2 P_n^2 + PQ\, P_n'^2}};$$

hence for given k, only those values of τ for which

$$\Delta = n^2 P_n^2 + PQ\, P_n'^2 - k^2 > 0 \quad \text{and} \quad \tau_t \leqq \tau \leqq 1 \tag{26}$$

will give real values of ϑ.

With n fixed, we investigate the region in the k, τ-plane for which (26) is satisfied. Certainly the region is bounded by the lines $\tau = \tau_t$ and $\tau = 1$. Differentiating $k^2 = n^2 P_n^2 + PQ P_n'^2$ with respect to τ and making use of Chaplygin's equation (I.1) in the form $PQ P_n'' + PQ' P_n' + n^2 P_n = 0$, we have

$$\frac{d(k^2)}{d\tau} = \frac{d}{d\tau}(n^2 P_n^2 + PQ P_n'^2) = P_n'^2 (P'Q - PQ') = -\frac{8\beta(2\beta+1)\tau^3}{[(2\beta+1)\tau - 1]^2} P_n'^2 < 0,$$

so that the boundary curve $k^2 = n^2 P_n^2 + PQ P_n'^2$ of the region (26) is monotonically decreasing with increasing τ in the range $\tau_t \leqq \tau \leqq 1$. A sketch of the region (26) for $k > 0$ is presented in Fig. 10.

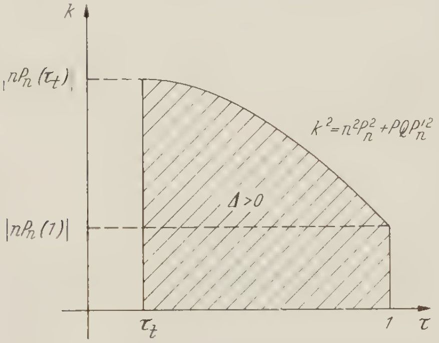

Fig. 10. Extent of the supersonic limit line in the hodograph plane for each k

From this graph we may deduce the extent of the supersonic limit line in the hodograph. Clearly for $k = 0$ the supersonic limit line extends from $\tau = \tau_t$ to $\tau = 1$. This remains true for values of k up to $k = |n P_n(1)|$. However as k increases from $|n P_n(1)|$ to $|n P_n(\tau_t)|$ the supersonic limit line detaches from the maximum circle $\tau = 1$ (while remaining attached to the sonic circle $\tau = \tau_t$) and shrinks towards the sonic circle. For values of k such that $|n P_n(1)| < k < |n P_n(\tau_t)|$ the supersonic limit line thus extends only up to a circle of radius τ_m, say, and $\tau_m \to \tau_t$ as $k \to |n P_n(\tau_t)|$. For $|n P_n(\tau_t)| < k$ no real branches of the supersonic limit line exist.

7. A flow with isolated sonic limit line

As a simple example of a flow with an isolated sonic limit line (see Section 6) we superpose the sonic limit line flow for $n = 1$ (see [3]) and the radial flow $n = 0$, *i.e.*

$$\psi = k\vartheta + \tau^{-\frac{1}{2}}[\varkappa + (1-\tau)^{\beta+1}] \sin\vartheta,$$

where

$$\varkappa = -2\left(\frac{2\beta}{2\beta+1}\right)^{\beta} \quad \left(= -1.267876 \text{ for } \gamma = \frac{7}{5}\right);$$

since $P_1(\tau_t) = -2\frac{(\beta+1)}{\sqrt{2\beta+1}}\left(\frac{2\beta}{2\beta+1}\right)^{\beta} < 0$, k must be chosen such that

$$|k| \geqq -P_1(\tau_t) = 2\frac{(\beta+1)}{\sqrt{2\beta+1}}\left(\frac{2\beta}{2\beta+1}\right)^{\beta}$$

(see (23)). In view of our earlier study of higher-order sonic limit singularities (Section I.4 (ii)) and the preceding section, we choose k to be exactly

$$k = 2\frac{(\beta+1)}{\sqrt{2\beta+1}}\left(\frac{2\beta}{2\beta+1}\right)^{\beta} \quad \left(= 1.811629 \text{ for } \gamma = \frac{7}{5}\right);$$

then

$$\psi_\vartheta = k + \tau^{-\frac{1}{2}}[\varkappa + (1-\tau)^{\beta+1}] \cos\vartheta,$$

$$\psi_{\vartheta\vartheta} = -\tau^{-\frac{1}{2}}[\varkappa + (1-\tau)^{\beta+1}] \sin\vartheta,$$

and the point $\tau = \tau_t$, $\vartheta = 0$ is a higher-order point on a sonic limit line at which $\psi_{\vartheta\vartheta} = 0$.

The coordinates in the physical plane are given by (*cf.* [3])

$$2x = \left[\varkappa \tau^{-1}(1-\tau)^{-\beta} + \beta + \tau^{-1}\right] \cos 2\vartheta + \\ + \varkappa \beta \int \tau^{-1}(1-\tau)^{-\beta-1}\, d\tau + (2\beta+1)\log\tau + 2k\tau^{-\frac{1}{2}}(1-\tau)^{-\beta}\cos\vartheta,$$

$$2y = \left[\varkappa \tau^{-1}(1-\tau)^{-\beta} + \beta + \tau^{-1}\right] \sin 2\vartheta - (\beta+1)\,(2\vartheta) + 2k\tau^{-\frac{1}{2}}(1-\tau)^{-\beta}\sin\vartheta,$$

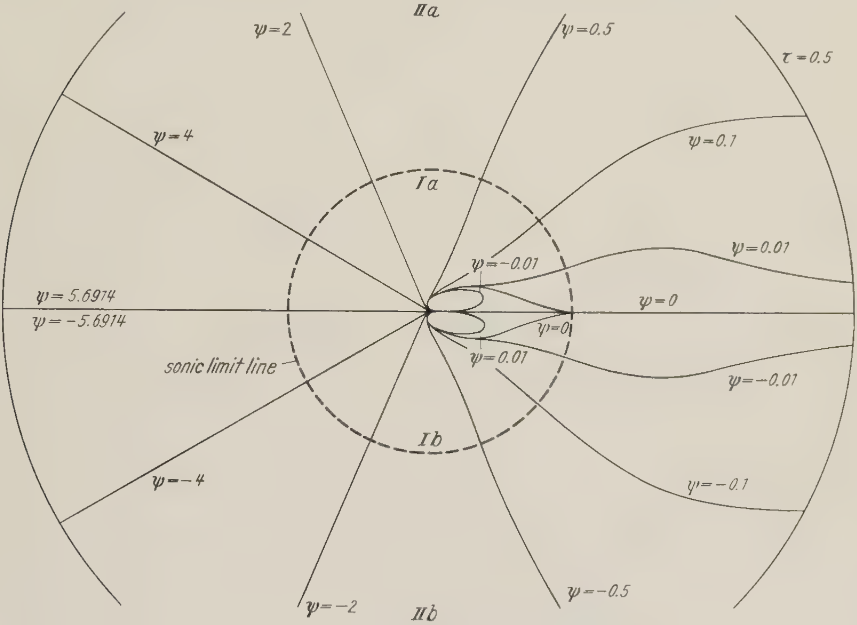

Fig. 11. Isolated sonic limit line. Hodograph plane

where the integral in the first equation can be evaluated in closed form for $\gamma = \frac{7}{5}$. The sonic limit line in the physical plane is then given by

$$x = (\beta+1)\cos\vartheta\left[2-\cos\vartheta\right] + \text{constant},$$

$$y = (\beta+1)\sin\vartheta\left[2-\cos\vartheta\right] - (\beta+1)\,\vartheta.$$

This curve is the locus of a point moving harmonically on the extended diameter of a circle, while the circle in turn rolls on a straight line. It is a trochoid-like curve and does not possess any cusps (*cf.* Section I.4 (ii)).

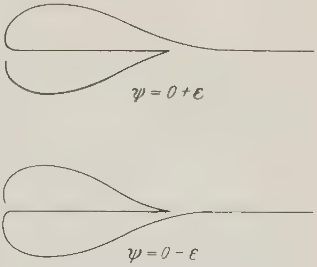

Fig. 12. Limiting behavior of the streamline $\psi = 0$

The flow in the hodograph plane, and hence also in the physical plane, is of quite a different character from that of the sonic limit line flows discussed previously.

In the hodograph plane the streamlines no longer form lemniscate-like loops inside a circular sector but behave as shown in Fig. 11. For a given value of $\psi = \psi_0 > 0$, the streamline in the hodograph consists of two portions: (*i*) one above the horizontal axis which meets the sonic limit line and ranges from $\tau = 0$ through all subsonic and all supersonic values of τ to the maximum value $\tau = 1$, and (*ii*) one below the horizontal axis which starts at $\tau = 0$ and loops around back to $\tau = 0$ without attaining supersonic speeds. The two portions which make up the streamline $\psi = -\psi_0 < 0$ form the symmetric image of $\psi = \psi_0$ in the axis $\vartheta = 0$. This explains the symmetry of the figure. The streamline $\psi = 0$ falls into the above pattern in a limiting sense (see Fig. 12); it degenerates into the straight segment $\vartheta = 0$ plus a symmetric cusped curve which passes through the higher-order limit point $\tau = \tau_t$, $\vartheta = 0$ (*cf.* Theorem I.13).

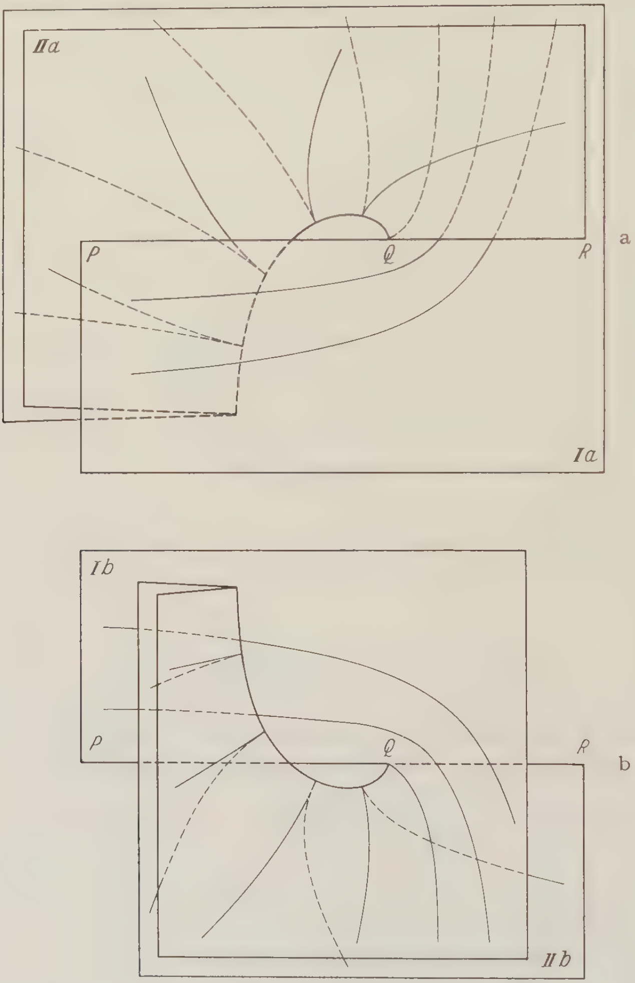

Fig. 13a and b. Isolated sonic limit line. Physical plane (schematic)

The corresponding streamlines in the physical plane therefore consist of subsonic-supersonic portions which are cusped at the sonic limit line, and smooth, purely subsonic portions.

To simplify the discussion of the streamlines in the physical plane we restrict ourselves first to that portion of the plane which corresponds to the upper half

$(0 \leqq \vartheta \leqq \pi)$ of the hodograph plane. In this range lie the cusped portions of some streamlines $(\psi \geqq 0)$ and the smooth portions of the rest. Thus we expect the streamlines in the physical plane corresponding to this half of the hodograph to lie on two sheets (a subsonic and a supersonic sheet), some entirely on the subsonic sheet and the others partially on both sheets. From this and the variations of ϑ along the streamlines in the hodograph we infer Fig. 13a. Note that parts of the subsonic sheet overlap in a manner similar to that in the examples of Section 5. By symmetry the portion of the sheeted surface corresponding to the lower half of the hodograph is as shown in Fig. 13b.

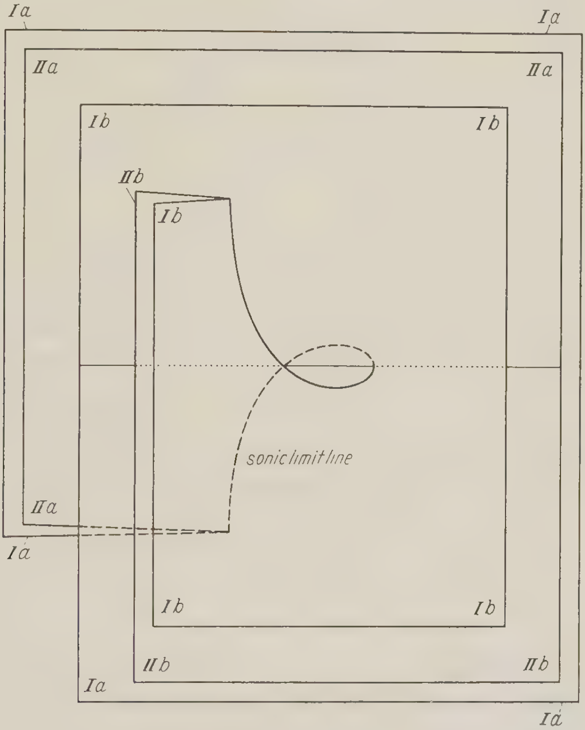

Fig. 14. Isolated sonic limit line. Sketch of multi-sheeted surface corresponding to physical plane

To obtain the complete picture we must join the two parts depicted in Figs. 13a and 13b along the line of symmetry PQR; the result is Fig. 14. Note that the entire surface consists of only two sheets — a subsonic and a supersonic sheet. However, the individual sheets overlap in parts, so that there are regions of single-, triple-, and quintuple-valuedness in parts of the x, y-plane.

This research was supported in part by the United States Air Force under Contract No. AF 49(638)-154, monitored by the AF Office of Scientific Research of the Air Research and Development Command, and in part by the Office of Ordnance Research, U. S. Army, under Contract DA-36-034-ORD-1486 with the University of Maryland. In 1957—1958 LUDFORD was a fellow of the John Simon Guggenheim Memorial Foundation.

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(Received May 21, 1958)

On Exterior Boundary Value Problems in Linear Elasticity

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Communicated by E. STERNBERG

1. Introduction

The questions considered in this note were suggested by a paper by FINN & NOLL on the uniqueness and non-existence of Stokes flow [*1*]. The equations of linear isotropic elasticity are only slightly more complicated than the equations governing Stokes flow, and it proves possible to extend the treatment of FINN & NOLL to analogous elastic problems.

The equations for the displacement field $\boldsymbol{q}$ of a linearly elastic body at rest and under no body forces may be written (*cf.* [2]) in the form

$$\Delta \boldsymbol{q} = \operatorname{grad} p, \qquad \operatorname{div} \boldsymbol{q} = -c p. \tag{1.1}$$

Here c is the positive constant $c = 1 - 2\sigma$, where σ is POISSON's ratio.

If we set $c=0$ in (1.1) and interpret $\boldsymbol{q}$ as the velocity field, then (1.1) are the equations of Stokes flow. The results of [*1*] may be phrased this way: Let the region of flow be bounded internally by closed surfaces (curves) $\mathfrak{B}$. We seek solutions of (1.1) (with $c=0$) which, in three dimensions, tend uniformly to zero at infinity or, in two dimensions, are bounded at infinity. There is at most one solution $\boldsymbol{q}$ of this type which assumes a given constant value $\boldsymbol{q} = \boldsymbol{q}_0$ at the internal boundary $\mathfrak{B}$. We note that this theorem remains valid also if the prescribed boundary value $\boldsymbol{q} = \boldsymbol{q}_0$ is not constant, although a simple physical interpretation for $\boldsymbol{q}$ as the flow past a rigid body can be given only when $\boldsymbol{q}_0$ is constant.

In this paper the corresponding theorem is proved for the case when $c > 0$ in (1.1). In this way, we get three different uniqueness theorems for boundary value problems in elasticity:

(a) Consider an infinite elastic medium bounded internally by closed surfaces $\mathfrak{B}$. There is at most one displacement $\boldsymbol{q}$ of the medium which assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at the internal boundaries $\mathfrak{B}$ and which tends uniformly to zero at infinity.

(b) Consider an infinite elastic medium bounded internally by an infinite cylinder $\mathfrak{B}$ (not necessarily circular). There is at most one plane displacement $\boldsymbol{q}$ of the medium in the plane perpendicular to the cylinder which assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at $\mathfrak{B}$ and which is bounded.

(c) Consider a thin infinite elastic plate bounded internally by closed contours $\mathfrak{B}$. There is at most one displacement $\boldsymbol{q}$ in the plane of the plate which assumes prescribed values $\boldsymbol{q}=\boldsymbol{q}_0$ at $\mathfrak{B}$ and which is bounded. All three results are special cases of Theorem 1, proved in Section 2. The proposition (b) corresponds to the case of plane strain. (c) corresponds to a state of generalized plane stress. In this case, $\boldsymbol{q}$ must be interpreted as the average displacement, the average being taken over the thickness of the plate. As is well known [*3*, p. 208], $\boldsymbol{q}$ then satisfies the two-dimensional form of the equations (1.1) when Poisson's ratio σ is replaced by $\sigma'=\frac{\sigma}{2-\sigma}$. Thus, in this case, the constant c must be given the positive value $c=1-2\sigma'=\frac{2-3\sigma}{2-\sigma}$. We also prove a uniqueness theorem for the transverse deflection w of an infinite thin plate bounded internally by closed curves $\mathfrak{B}$. The function w is biharmonic: $\Delta\Delta w=0$. We show that, if the plate is clamped in a prescribed way at $\mathfrak{B}$, so that w and $\partial w/\partial n$ are prescribed at $\mathfrak{B}$, then there is at most one deflection w which is of the order $O(r)$ as $r\to\infty$.

The essential difference between the results obtained here and the uniqueness theorems previously known lies in the conditions at infinity. Previously, complicated regularity conditions at infinity had to be imposed on $\boldsymbol{q}$ or w. We merely require simple limit conditions. It should not be difficult to obtain similar results when the surface tractions, rather than the displacements, are prescribed at the interior boundaries.

As in the case of Stokes flow so also in elasticity we observe the striking difference between the situations in two and three dimensions. In three dimensions we get a unique solution if we require that $\boldsymbol{q}(\infty)=0$, *i.e.* that the "boundary at infinity" is fixed. If we impose the same condition in two dimensions there is, in general, no solution at all. For a physical interpretation consider, for example, a very large thin plate with a fixed external boundary. At first glance our theorem seems to imply that it is not possible to impose a rigid displacement $\boldsymbol{q}=\boldsymbol{q}_0=\text{const.}$ on the internal boundaries, a result that contradicts physical experience. However, the correct interpretation is the following: The gradient of the displacement $\boldsymbol{q}$, and hence the strain, is very small everywhere in the plate. In the limit as the external boundary expands to infinity the gradient of the displacement tends to zero. The displacement $\boldsymbol{q}$ converges to its internal boundary value $\boldsymbol{q}_0$. But this convergence is not uniform, and the condition that $\boldsymbol{q}=0$ on the external boundary is not preserved in the limit. A similar explanation must be given in the case of the transverse deflection of a large plate. If the external boundary is clamped ($w=\partial w/\partial n=0$) we get a unique solution. In the limit, however, the conditions $w=\partial w/\partial n=0$ at the "boundary at infinity" get lost.

If we impose a displacement $\boldsymbol{q}=\boldsymbol{q}_0$ on the boundary of an internal cavity in a large elastic body with fixed external boundary, we shall get a displacement $\boldsymbol{q}$ which approximates the solution for the infinite medium determined uniquely by the condition $\boldsymbol{q}(\infty)=0$. This solution will be given explicitly in Section 4 for $\boldsymbol{q}_0=\text{const.}$ and a spherical cavity. The displacement gradient is not small, and the strains and stresses will remain finite in the limit as the body becomes infinite. In this case, the boundary condition $\boldsymbol{q}=0$ on the external boundary is preserved in the limit.

2. The basic lemmas

Let $\boldsymbol{q}$ be a vector field defined and three times continuously differentiable in a neighborhood of infinity.

Lemma 1. In three dimensions, assume that $\boldsymbol{q}=\boldsymbol{q}_0+o(1)$. Then

$$\Delta(\operatorname{curl}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{curl}\boldsymbol{q}=O(r^{-2}), \tag{2.1}$$

and

$$\Delta(\operatorname{div}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{div}\boldsymbol{q}=O(r^{-2}). \tag{2.2}$$

Lemma 2. In two dimensions, assume that $\boldsymbol{q}$ is bounded. Then

$$\Delta(\operatorname{curl}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{curl}\boldsymbol{q}=O(r^{-1}), \tag{2.3}$$

and

$$\Delta(\operatorname{div}\boldsymbol{q})=0 \quad \text{implies} \quad \operatorname{div}\boldsymbol{q}=O(r^{-1}). \tag{2.4}$$

The statements concerning the curl are Lemmas 4 and 5 in [*1*]. The propositions concerning the divergence can be proved in precisely the same manner as these lemmas.

Lemma 3. Assume that w is a scalar field defined and biharmonic in a neighborhood of infinity. Then $w=O(r)$ implies that grad w is bounded.

Proof. We denote the mean value of a function w over the circumference of a circle of radius r with center at Q by $m_0(w,r,Q)$. If w is biharmonic in a region containing this circle we have the following mean value theorem (*cf.* [*4*], p. 316, (11)*):

$$3w(Q)=4m_0(w,\tfrac{1}{2}r,Q)-m_0(w,r,Q). \tag{2.5}$$

We may replace w by grad w in (2.5) because grad w is biharmonic if w is. If we do so, multiply then by r, integrate between $r=0$ and r and finally apply GREEN's theorem, we get

$$3r\operatorname{grad}w(Q)=8\boldsymbol{m}_1(w,\tfrac{1}{2}r,Q)-\boldsymbol{m}_1(w,r,Q), \tag{2.6}$$

where

$$\boldsymbol{m}_1(w,r,Q)=\frac{1}{\omega}\int_\omega w(Q+r\boldsymbol{e})\,\boldsymbol{e}\,d\omega.$$

We have

$$|\boldsymbol{m}_1(w,r,Q)|\leqq\max_P|w(P)|,$$

where P varies over the disc of radius r around Q. Hence, by (2.6)

$$r\,|\operatorname{grad}w(Q)|\leqq 3\max_P|w(P)|.$$

Applying this inequality to circles of increasing radius with centers that tend to infinity, we obtain the assertion of the lemma.

---

* It is proved there in the case of three dimensions, but both the proof and the result are applicable in any dimension. The lemma stated here is also valid in higher dimensions.

3. Uniqueness of the exterior boundary value problems

Theorem 1. Let $\mathfrak{V}$ be a region in space (in the plane) exterior to an internal boundary $\mathfrak{B}$ consisting of a finite number of piecewise smooth closed surfaces (curves). Then there is at most one vector field $\boldsymbol{q}$, defined and twice continuously differentiable in $\mathfrak{V}$, with the following properties:

(a) $\boldsymbol{q}$ is a solution of the equations

$$\Delta \boldsymbol{q} = \operatorname{grad} p, \qquad \operatorname{div} \boldsymbol{q} = -c p, \tag{3.1}$$

where c is a given positive constant.

(b) $\boldsymbol{q}$ is continuous up to $\mathfrak{B}$ and assumes prescribed values

$$\boldsymbol{q} = \boldsymbol{q}_0 \quad \text{at } \mathfrak{B}. \tag{3.2}$$

(c) In three dimensions $\boldsymbol{q}$ converges uniformly to zero at infinity: $\boldsymbol{q}(\infty) = 0$ (in two dimensions, $\boldsymbol{q}$ is bounded).

Proof. We observe that (3.1) implies

$$\operatorname{curl}\operatorname{curl} \boldsymbol{q} = -(1+c)\operatorname{grad} p. \tag{3.3}$$

Operating on (3.3) with curl and div yields

$$\Delta(\operatorname{curl} \boldsymbol{q}) = 0, \qquad \Delta p = 0. \tag{3.4}$$

Here we have made use of the fact that $\boldsymbol{q}$ has continuous third derivatives. It is shown in [2] that this is a consequence of (a).

Since the equations (3.1) are linear and homogeneous, it is sufficient to show that $\boldsymbol{q}_0 = 0$ implies $\boldsymbol{q} \equiv 0$. We do this in the three-dimensional case.

By Lemma 1 and (3.4) we have

$$\operatorname{curl} \boldsymbol{q} = O(r^{-2}), \qquad p = -\frac{1}{c}\operatorname{div} \boldsymbol{q} = O(r^{-2}). \tag{3.5}$$

We use the Green's identity

$$\int_{\mathfrak{R}} (\operatorname{curl} \boldsymbol{q})^2\, dV = \int_{\mathfrak{R}} (\boldsymbol{q}\cdot\operatorname{curl}\operatorname{curl} \boldsymbol{q})\, dV - \int_{\mathfrak{S}} \operatorname{curl} \boldsymbol{q}\cdot(\boldsymbol{q}\times\boldsymbol{n})\, dS, \tag{3.6}$$

valid for any smooth vector field defined in the finite region $\mathfrak{R}$ with piecewise smooth boundary $\mathfrak{S}$.

By (3.3) we have

$$\begin{aligned} \boldsymbol{q}\cdot\operatorname{curl}\operatorname{curl} \boldsymbol{q} &= -(1+c)\,\boldsymbol{q}\cdot\operatorname{grad} p \\ &= -(1+c)\left[\operatorname{div}(p\,\boldsymbol{q}) - p\operatorname{div} \boldsymbol{q}\right]. \end{aligned}$$

By use of $(3.1)_2$ and the divergence theorem, the identity (3.6), applied to the region $\mathfrak{V}_r$ between the internal boundary $\mathfrak{B}$ and a sphere $\mathfrak{S}_r$ of radius r containing $\mathfrak{B}$, gives

$$\int_{\mathfrak{B}_r} \left[(\operatorname{curl} \boldsymbol{q})^2 + \frac{1+c}{c}(\operatorname{div} \boldsymbol{q})^2\right] dV = -\int_{\mathfrak{S}_r} \left[(1+c)\,p\,\boldsymbol{q}\cdot\boldsymbol{n} + \operatorname{curl} \boldsymbol{q}\cdot(\boldsymbol{q}\times\boldsymbol{n})\right] dS. \tag{3.7}$$

Here we have used the fact that $\boldsymbol{q} = \boldsymbol{q}_0 = 0$ on $\mathfrak{B}$. It follows from (3.5) and $\boldsymbol{q} = o(1)$ that the right-hand side of (3.3) tends to zero as $r \to \infty$.

Hence

$$\int_{\mathfrak{B}} \left[(\operatorname{curl} \boldsymbol{q})^2 + \frac{1+c}{c} (\operatorname{div} \boldsymbol{q})^2\right] dV = 0.$$

Since the integrand is non-negative and continuous, it must vanish. Hence $\operatorname{curl} \boldsymbol{q} = 0$, $\operatorname{div} \boldsymbol{q} = 0$ whence $\Delta \boldsymbol{q} = 0$. Since $\boldsymbol{q} = 0$ on $\mathfrak{B}$ and $\boldsymbol{q} = o(1)$ the maximum and minimum principle for harmonic functions implies $\boldsymbol{q} \equiv 0$, *q.e.d.*

The proof for the two-dimensional case is the same as that of Theorem II in [*1*] when modified along the lines indicated above.

Theorem 2. Let $\mathfrak{B}$ be a region in the plane exterior to an internal boundary $\mathfrak{B}$ consisting of a finite number of piecewise smooth curves. Then there is at most one scalar field w with the following properties:

(a) w is defined and biharmonic in $\mathfrak{B}$: $\Delta \Delta w = 0$.

(b) w and its first partial derivatives are continuous up to $\mathfrak{B}$ and w and $\partial w / \partial n$ assume prescribed values

$$w = w_0, \qquad \frac{\partial w}{\partial n} = w_1 \quad \text{at } \mathfrak{B}.$$

(c) $w = O(r)$ as $r \to \infty$.

Proof. We may regard w as the stream function of a plane viscous flow $\boldsymbol{q}$. Then

$$\boldsymbol{q} = (\operatorname{grad} w)^{\perp},$$

where $\perp$ indicates the operation of rotating a vector counterclockwise by a right angle. The condition (b) implies that $\operatorname{grad} w$, and hence $\boldsymbol{q}$ assumes prescribed values $\boldsymbol{q} = \boldsymbol{q}_0$ at $\mathfrak{B}$. By Lemma 3 it follows from (c) that $\boldsymbol{q}$ is bounded. The results of [*1*] imply that $\boldsymbol{q}$, and hence $\operatorname{grad} w$, is uniquely determined. This implies the uniqueness of w because the boundary values $w = w_0$ at $\mathfrak{B}$ are given.

4. A representation formula

The representation theorem (Theorem III) of [*1*] may be extended to the case of the three-dimensional equation (1.1) in the following way: Let $\boldsymbol{q}$ be a solution of (1.1) defined in a neighborhood of infinity $\mathfrak{E}$ which is star-shaped relative to a point Q. Then there is a harmonic vector-field $\boldsymbol{u}$, defined in $\mathfrak{E}$ and harmonic at infinity, such that

$$\boldsymbol{q} = \boldsymbol{u} + \operatorname{grad} \psi, \tag{4.1}$$

where

$$\psi = \psi(Q + r\,\boldsymbol{e}) = \frac{1}{4(1+c)} r^{\frac{1}{2}} \int_0^r s^{\frac{1}{2}} \operatorname{div} \boldsymbol{u}(Q + s\,\boldsymbol{e})\, ds. \tag{4.2}$$

The proof is identical to the one given in [*1*]*. Using the representation (4.1), we may obtain the displacement $\boldsymbol{q}$ caused by a rigid translation $\boldsymbol{q} = \boldsymbol{q}_0 = \text{const.}$ of a spherical cavity in an infinite elastic medium. We try

$$\boldsymbol{u}(Q + r\,\boldsymbol{e}) = \alpha \frac{1}{r} \boldsymbol{q}_0 + \beta \frac{1}{r^3} \left[\boldsymbol{q}_0 - 3(\boldsymbol{q}_0 \cdot \boldsymbol{e})\, \boldsymbol{e}\right], \tag{4.3}$$

---

* The equations (4.3), (4.4) and (6.2) in [*1*] contain a mistake in sign.

which is harmonic at infinity. Let Q be the center of the sphere and a its radius. With (4.1) and (4.2) we can determine the constants α and β so that the boundary condition $\boldsymbol{q}(Q+a\boldsymbol{e})=\boldsymbol{q}_0$ is satisfied. The result of the calculation is $(\boldsymbol{r}=r\boldsymbol{e})$

$$\boldsymbol{q}=\frac{1}{2(2+3c)}\left\{\boldsymbol{q}_0\left[3(1+2c)\frac{a}{r}+\left(\frac{a}{r}\right)^3\right]+3\boldsymbol{r}(\boldsymbol{q}_0\cdot\boldsymbol{r})\left[\frac{a}{r^3}-\frac{a^3}{r^5}\right]\right\}. \tag{4.4}$$

For $c=0$ this reduces to the classical solution of Stokes for the flow past a sphere.

Acknowledgement. This work was supported by Contract AF 18(600)-1138 of the Air Force Office of Scientific Research and Contract DA-36-061-ORD-490 of the Office of Ordnance Research, U. S. Army.

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(Received July 1, 1958)

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Druck der Universitätsdruckerei H. Stürtz AG., Würzburg